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Flocking of multi-agent systems with a dynamic virtual leader

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This paper considers the flocking problem of a group of autonomous agents moving in Euclidean space with a virtual leader. We investigate the dynamic properties of the group for the case where the state of the virtual leader may be time-varying and the topology of the neighbouring relations between agents is dynamic. To track such a leader, we introduce a set of switching control laws that enable the entire group to generate the desired stable flocking motion. The control law acting on each agent relies on the state information of its neighbouring agents and the external reference signal (or 'virtual leader'). Then we prove that, if the acceleration of the virtual leader is known, then each agent can follow the virtual leader, and the convergence rate of the centre of mass (CoM) can be estimated; if the acceleration is unknown, then the velocities of all agents asymptotically approach the velocity of the CoM, thus the flocking motion can be obtained. However, in this case, the final velocity of the group may not be equal to the desired velocity. Numerical simulations are worked out to illustrate our theoretical results.

Keywords: collective dynamic behaviour; flocking; multiagent systems; swarm intelligence; virtual leader

1. Introduction

Flocking is ubiquitous in nature, e.g., flocking of birds, schooling of fish, herding of animals, and swarming of bacteria, and it is a form of collective behaviour of multiple interacting agents. In recent years, there has been increasing research interest in the distributed control/coordination of the motion of multiple dynamic agents/robots and the control design of multi-agent systems. In a multi-agent system, agents are usually coupled and interconnected with some simple rules. Understanding the mechanisms and operational principles in them can provide useful ideas for developing formation control, distributed cooperative control and coordination of multiple mobile autonomous agents/robots. Applications of multi-agent systems can be found in many areas, such as biology(e.g., aggregation behaviour of animals), physics (e.g., collective motion of particles), robotics and control engineering (e.g., formation control of robots, cooperative control of unmanned aerial vehicles (UAVs), scheduling of automated highway systems, coordination/formation of underwater vehicles and attitude alignment of satellite clusters). There has been considerable effort in modelling and exploring the collective dynamics, and trying to understand how a group of autonomous creatures or man-made mobile autonomous agents/robots can cluster in formations without centralised coordination and control (Reynolds 1987; Vicsek, Czirok, Ben-Jacob, Cohen and Shochet 1995; Leonard and Fiorelli 2001; Gazi and Passino 2003; Jadbabaie, Lin and Morse 2003; Tanner, Jadbabaie and Pappas 2003, 2007; Fax and Murray 2004; Lin, Broucke and Francis 2004; Olfati-Saber and Murray 2004; Savkin 2004; Shi, Wang, Chu and Xu 2005; Chu, Wang, Chen and Mu 2006; Hong, Hu and Gao 2006; Olfati-Saber 2006; Shi, Wang and Chu 2006a, Shi, Wang, Chu and Xiao 2006b). Many results have been obtained with local rules applied to each agent in a considered multi-agent system.

Stimulated by the simulation results in Reynolds (1987), Tanner, Jadbabaie and Pappas (2003, 2007) considered a group of mobile agents moving in the plane with double-integrator dynamics. They introduced a set of control laws that enable the group to generate stable flocking motion, but these control laws cannot regulate the final speed and heading of the group. Due to the fact that in some cases, the regulation of agents has certain purposes such as achieving desired common speed and heading, or arriving at a desired destination, the cooperation/coordination of multiple mobile agents with some

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virtual leaders is an interesting and important topic. There have been some papers dealing with this issue in the literature. For example, Leonard and Fiorelli (2001) viewed reference points as virtual leaders for manipulating the geometry of an autonomous vehicle group and directing the motion of the group; Olfati-Saber (2006) used virtual leaders to accomplish obstacle avoidance; Shi et al. (2006a) regarded the reference signal as a virtual leader for guiding the agent group to move at the desired constant velocity. In Olfati-Saber (2006) and Shi et al. (2006a), the authors used different moving reference frames to analyse the flocking problem. Olfati-Saber (2006) chose the position of the centre of mass (CoM) of the group as the origin of the moving frame, whereas Shi et al. (2006a) chose the position of the virtual leader as the origin of the moving frame. The difference between these two approaches will be discussed in detail in the paper.

In this paper, we consider the flocking problem of multiple mobile autonomous agents moving in an n-dimensional Euclidean space with point mass dynamics. We divide the considered problem into two issues, and propose the corresponding control laws and develop Lyapunov-based approach to analyse them. First, by viewing the external reference signal as a virtual leader, we show that, if all agents know the velocity and acceleration of the virtual leader, then they eventually move ahead at the desired velocity and maintain constant distances between them (called 'strong flocking'). This means that all agents can track the virtual leader. Secondly, for the case where we let all agents move ahead at a common velocity and do not care about the explicit value (called 'weak flocking'), we prove that the weak flocking can be achieved if all agents only know the velocity of the virtual leader. The flocking problem is analysed in different cases and the main results are listed in a table in § 5.

This paper is organised as follows. In § 2, we formulate the problem to be investigated. Then we design a set of local control laws and analyse the stability of the resulting closed-loop system in § 3. Section 4 gives numerical simulations to illustrate the theoretical results. Finally, we briefly summarise our results in § 5.

2. Problem formulation

In this paper, we consider a group of N agents moving in an n-dimensional Euclidean space; each has point mass dynamics described by

$$\left.\begin{aligned} \dot{x}^i &= v^i, \\ m_i\dot{v}^i &= u^i - k_i v^i, \quad i = 1, \ldots, N, \end{aligned}\right\} \tag{1}$$

where $x^i \in \mathbb{R}^n$ is the position vector of agent i, $v^i \in \mathbb{R}^n$ is its velocity vector, $m_i > 0$ is its mass, $u^i \in \mathbb{R}^n$ is the control input acting on agent $i, k_i > 0$ is the 'velocity damping gain', and $-k_i v^i$ is the velocity damping term. Here we assume that the damping force is in proportion to the magnitude of velocity and the damping gains k_i, $i = 1, \ldots, N$, are not equal to each other.

Our main objective is to study the flocking problem of how to make the entire group move at a desired velocity and maintain constant distances between the agents. The desired velocity is supposed to be a time-varying and smooth function, which means that the state of the virtual leader keeps changing. In order to achieve our goal, we try to regulate each agent velocity to the desired velocity, reduce the velocity differences between neighbouring agents, regulate their distances such that their potentials become minima, and at the same time, compensate for the velocity damping. Hence, we choose the control law u^i for agent i to be

$$u^i = \alpha^i + \beta^i + \gamma^i + \delta^i, \tag{2}$$

where α^i is used to regulate the potential of agent i to its minimum, β^i is used to regulate the velocity of agent i to the weighted average of the velocities of its neighbours, γ^i is used to regulate the velocity of agent i to the desired velocity, and δ^i is used to compensate for the velocity damping (all to be designed later).

In the design of control law (2), α^i can be determined by an artificial social potential function, V^i, a function of the relative distances between agent i and its flockmates. Freedom from collisions and cohesion in the group can be guaranteed by this term. β^i can be obtained according to the alignment or velocity matching with neighbours among agents. γ^i is designed based on the external signal, i.e., the desired velocity and acceleration. δ^i can be simply taken to be the reverse of the damping force.

We also consider the weak case of flocking problem. Namely, in case the acceleration of the virtual leader is not available to each agent, then we can only design a control law to make all agents move with a common velocity. However the velocity value cannot be estimated explicitly.

Definition 1 (neighbouring graph)**:** The neighbouring graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, is an undirected graph consisting of a set of vertices, $\mathcal{V} = \{n_1, \ldots, n_N\}$, indexed by the agents in the group, and a set of edges, $\mathcal{E} = \{(n_i, n_j) \in \mathcal{V} \times \mathcal{V} \mid \|x^i - x^j\| \leq R\}$, containing unordered pairs of vertices that represent the neighbouring relations, where $R > 0$ is a constant.

In the following we make use of the neighbouring graph to describe the sensor information flow in the group. In $\mathcal{G}$, an edge (n_i, n_j) means that agent i can sense agent j, and it will regulate its state based on the position and velocity of agent j. In this paper, we mainly consider the dynamic and symmetric neighbouring relations between agents. Let $\mathcal{I} = \{1, \ldots, N\}$. We let $\mathcal{N}_i \triangleq \{j \mid \|x^{ij}\| \le R\} \subseteq \mathcal{I} \backslash \{i\}$ be the set containing all neighbours of agent i, where $x^{ij} = x^i - x^j$ denotes the relative position vector between agents i and j; $R > 0$ can be viewed as the sensing radius of the sensors. Here we assume that the sensors of all agents have the same sensing range. During the course of motion, the relative distances between agents vary with time, so the neighbours of each agent are not fixed, which generates the switching neighbouring graph $\mathcal{G}_\sigma$, where σ is a switching signal and is a piecewise constant function $\sigma(t): [0, +\infty) \to \mathcal{P}, \mathcal{P}$ is a finite index set where the number of the indices is equal to the number of the connected graph $\mathcal{G}_\sigma$ in the group. The switching signal σ relies on the distances between agents according to the local sensing range of each agent. In the discussion to follow, we assume that the neighbouring graph $\mathcal{G}_\sigma$ remains connected, which ensures that the group will not be divided into several isolated subgroups. In order to depict the potential between the agents, we present the following definition.

Definition 2 (Tanner et al. 2007) (potential function)**:** Potential V^{ij} is a differentiable, nonnegative function of the distance $\|x^{ij}\|$ between agents i and j, such that $V^{ij}(\|x^{ij}\|) \to +\infty$ as $\|x^{ij}\| \to 0$; V^{ij} attains its unique minimum when agents i and j are located at a desired distance; $(d/d\|x^{ij}\|)V^{ij}(\|x^{ij}\|) = 0$ if $\|x^{ij}\| > R$.

Functions $V^{ij}, i, j = 1, \ldots, N$, are the artificial social potential functions that govern the interindividual interactions. By the definition of V^{ij}, we obtain that if $\|x^{ij}\| > R$, then $V^{ij}(\|x^{ij}\|) = V^{ij}(R)$. One example of such potential functions is the following:

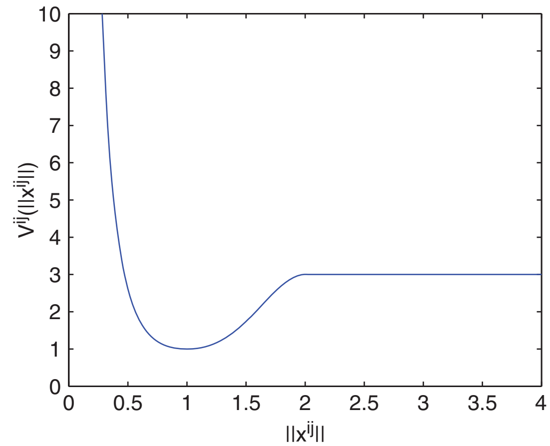

Figure 1. Potential function V^{ij}.

By the definition of V^{ij}, the total potential of agent i can be expressed as

$$V^i = \sum_{j \in \mathcal{N}_i} V^{ij}(\|x^{ij}\|) + \sum_{j \notin \mathcal{N}_i, j \ne i} V^{ij}(R).$$

Certainly, in reality, according to different cases, we can define different interaction potential functions such as the functions considered in Leonard and Fiorelli (2001), Gazi and Passino (2003) and Olfati-Saber (2006).

3. Main results

In this section, we first present explicit formulation for the terms α^i, β^i, γ^i, and δ^i in the control law (2), then we investigate the stability properties of the closed-loop system by employing matrix analysis, algebraic graph theory and nonsmooth analysis. For basic concepts and results, we refer to Clarke (1983); Horn and Johnson (1985); Paden and Sastry (1987); Shevitz and Paden (1994) and Godsil and Royle (2001).

$$V^{ij}(\|x^{ij}\|) = \begin{cases} a \ln \|x^{ij}\|^2 + \dfrac{b}{\|x^{ij}\|^2}, & 0 < \|x^{ij}\| \le \sqrt{b/a}; \\ a[1 + \ln(b/a)] + \cos\left[\left(1 + \dfrac{\|x^{ij}\|^2 - (b/a)}{R^2 - (b/a)}\right)\pi\right] + 1, & \sqrt{b/a} < \|x^{ij}\| \le R; \\ a[1 + \ln(b/a)] + 2, & \|x^{ij}\| > R, \end{cases} \tag{3}$$

where a, b, and R are some positive constants such that $b > a/e$ and $R > \sqrt{b/a}$. Note that the assumption $R > \sqrt{b/a}$ is reasonable as this implies that the desired distance between two agents is smaller than the agent's sensing range R. It is easy to see that V^{ij} attains its unique minimum $a[1 + \ln(b/a)]$ when $\|x^{ij}\| = \sqrt{b/a}$. Figure 1 depicts the curve of one potential function, where $a = 1, b = 1$, and $R = 2$.

We take the control law u^i for agent i to be

$$u^i = \underbrace{-\sum\nolimits_{j \in \mathcal{N}_i} \nabla_{x^i} V^{ij}}_{\alpha^i} \underbrace{-\sum\nolimits_{j \in \mathcal{N}_i} w_{ij}\left(v^i - v^j\right)}_{\beta^i} \underbrace{-h_s^i m_i\left(v^i - v^0(t)\right) + g_i m_i a_0(t)}_{\gamma^i} + \underbrace{k_i v^i}_{\delta^i}, \tag{4}$$

where $w_{ij} \geq 0$ and $w_{ii} = 0$, $i, j = 1, \ldots, N$, represent the interaction coefficients; $v^0(t) \in \mathbb{R}^n$ is the desired velocity and $\dot{v}^0(t) = a_0(t)$; $h_s^i \geq 0$ represents the intensity of influence of the reference signal on the motion of agent i, $h_s^i = h_i$ if agent i knows the desired velocity, where $h_i > 0$ is a constant, and is 0 otherwise; $g_i = 1$ if agent i knows the acceleration $a_0(t)$ and is 0 otherwise. $w_{ij} = c_{ij}$ is fixed if agent j is a neighbour of agent i, where $c_{ij} > 0$ ($\forall i \neq j$) is a constant, and is 0 otherwise. Here we always assume that $c_{ij} = c_{ji}$, which means that the interaction between agents is reciprocal. We write $W_\sigma = [w_{ij}]_\sigma \in \mathbb{R}^{N \times N}$ for the interaction coefficient matrix (or 'coupling matrix'). Thus, W_σ is always symmetric, and by the assumption of the connectivity of the neighbouring graph $\mathcal{G}_\sigma$, W_σ is always irreducible. During the course of motion, each agent regulates its position and velocity based on the external signal and the state information of its neighbours. However, it is known that, in reality, because of the influence of some external factors, the reference signal is not always detected by all agents in the group. In this paper, we will consider the case where the signal is sent continuously and at any time, there exists at least one agent in the group who can detect it, i.e., there exists $h_s^k = h_k > 0$ for some $k \in \mathcal{I}$, where s is a switching signal and is a piecewise constant function $s(t)\colon [0, +\infty) \to \widehat{\mathcal{P}}$, $\widehat{\mathcal{P}}$ is a finite index set where the number of the indices is equal to $2^N - 1$. Notice that the switching signals s and σ in the switching neighbouring graph $\mathcal{G}_\sigma$ may be different, so we combine the two switching signals into one switching signal $\overline{\sigma}(t)\colon [0, +\infty) \to \overline{\mathcal{P}}$, $\overline{\mathcal{P}}$ is a finite index set where the number of the indices is equal to the number of the elements of set $\{(x, y) \mid \forall x \in \mathcal{P}, \forall y \in \widehat{\mathcal{P}}\}$.

Since the neighbour set $\mathcal{N}_i$ of each agent i is time-varying and the agents who know the external signal at different times are different, then discontinuities will be introduced to the right hand side of control law (4). Hence, the agent dynamics is expressed in terms of differential inclusions

$$\left.\begin{aligned} \dot{x}^i &= v^i, \\ m_i \dot{v}^i &\in^{a.e.} K\big[u^i - k_i v^i\big], \quad i = 1, \ldots, N, \end{aligned}\right\} \tag{5}$$

where $K[\cdot]$ is a differential inclusion, and *a.e.* stands for 'almost everywhere'.

3.1 *Flocking control: group motion with desired velocity*

In this case, we assume that all agents know the desired velocity $v^0(t)$ and the acceleration $a_0(t)$, i.e., $h_s^i = h_i$ and $g_i = 1$ for all $i \in \mathcal{I}$. However, not all coefficients h_i, $i = 1, \ldots, N$, are considered in the design of the control law. In the following, we assume that, at any time, there exists at least one coefficient h_s^k, $k \in \mathcal{I}$, to be positive in control law (4). Then we have the following results.

3.1.1 *Stability analysis*

Theorem 1: *Taking the control law in* (4), *all agent velocities in the group described in* (5) *asymptotically approach the desired velocity, avoidance of collisions between the agents is ensured, and the group final configuration minimises all agent potentials.*

This theorem becomes clear after Theorem 2 is proved, so we proceed to present Theorem 2. We define the following error vectors:

$$e_p^i = x^i - \int_{t_0}^{t} v^0(\tau)\mathrm{d}\tau \quad \text{and} \quad e_v^i = v^i - v^0(t),$$

where t is the time variable and t_0 is the initial time. Then e_v^i represents the velocity difference vector between the actual velocity and the desired velocity of agent i. It is easy to see that $\dot{e}_p^i = e_v^i$ and $\dot{e}_v^i = \dot{v}^i - a_0(t)$. Hence, the error dynamics is given by

$$\left.\begin{aligned} \dot{e}_p^i &= e_v^i, \\ m_i \dot{e}_v^i &\in^{a.e.} K\big[u^i - k_i v^i - m_i a_0(t)\big], \quad i = 1, \ldots, N. \end{aligned}\right\} \tag{6}$$

Note that, in fact, we choose a moving reference frame and take the position of the virtual leader as the origin (called 'the first moving reference frame'). By the definition of V^{ij}, it follows that $V^{ij}(\|x^{ij}\|) = V^{ij}(\|e_p^{ij}\|) \triangleq \widetilde{V}^{ij}$, where $e_p^{ij} = e_p^i - e_p^j$, and hence $\widetilde{V}^i = V^i$ and $\nabla_{e_p^i} \widetilde{V}^{ij} = \nabla_{x^i} V^{ij}$. Moreover, by the symmetry of $\widetilde{V}^{ij}$ with respect to e_p^{ij} and by $e_p^{ij} = -e_p^{ji}$, it follows that $\nabla_{e_p^{ij}} \widetilde{V}^{ij} = \nabla_{e_p^i} \widetilde{V}^{ij} = -\nabla_{e_p^j} \widetilde{V}^{ij}$. Thus the control input for agent i in the error system (6) has the following form:

$$\begin{aligned} u^i = & \underbrace{-\sum\nolimits_{j \in \mathcal{N}_i} \nabla_{e_p^i} \widetilde{V}^{ij}}_{\alpha^i} \underbrace{-\sum\nolimits_{j \in \mathcal{N}_i} w_{ij}\big(e_v^i - e_v^j\big)}_{\beta^i} \\ & \underbrace{-h_{\overline{\sigma}}^i m_i e_v^i + m_i a_0(t)}_{\gamma^i} + \underbrace{k_i v^i}_{\delta^i}. \end{aligned} \tag{7}$$

Theorem 2: *Taking the control law in* (7), *all agent velocities in the system described in* (6) *asymptotically approach zero, avoidance of collisions between the agents is ensured, and the group final configuration minimises all agent potentials.*

Proof: Consider the following Lyapunov-like function:

$$J = \frac{1}{2}\sum_{i=1}^{N}\Big(\widetilde{V}^i + m_i e_v^{iT} e_v^i\Big).$$

It is easy to see that J is the sum of the total artificial potential energy and the total kinetic energy of all agents in the error system. Define the level set of J in the space of agent velocities and relative distances in the error system:

$$\Omega = \left\{ \left(e_v^i, e_p^{ij} \right) \mid J \le c, \ c > 0 \right\}.$$

Though the neighbouring relations vary with time, under the assumption of the connectivity of $\mathcal{G}_{\overline{\sigma}}$, the set Ω is compact. This is because the set $\{e_v^i, e_p^{ij}\}$ with $J \le c$ is closed by continuity. Moreover, boundedness can be proved by the connectivity of $\mathcal{G}_{\overline{\sigma}}$. More specifically, because $\mathcal{G}_{\overline{\sigma}}$ is always connected, there must be a path connecting any two agents i and j in the group and its length does not exceed $N-1$, and on the other hand, the distance between two interconnected agents is not more than R, hence, we have $\|e_p^{ij}\| \le (N-1)R$. By similar analysis, $e_v^{iT} e_v^i \le 2c/m_i$; thus $\|e_v^i\| \le \sqrt{2c/m_i}$. The invariance of Ω will be established in the sequel once J is shown to be non-increasing.

Function J is differentiable, but its derivative along the system's trajectories is not single-valued, for we do not know the value of $\dot{e}_v^i$ at the switching instants. We can only ensure that $m_i \dot{e}_v^i \in^{a.e.} K[u^i - k_i v^i - m_i a_0(t)]$. Since the potential function is differentiable at the transition point, it does not introduce discontinuities in the control law though the neighbouring graph is time-varying. Hence,

$$\begin{aligned} m_i \dot{e}_v^i &\in^{a.e.} K\left[-\sum_{j \in \mathcal{N}_i} \nabla_{e_p^i} \widetilde{V}^{ij} - \sum_{j \in \mathcal{N}_i} w_{ij}\left(e_v^i - e_v^j\right) - h_{\overline{\sigma}}^i m_i e_v^i \right] \\ &\subseteq -\sum_{j \in \mathcal{N}_i} \nabla_{e_p^i} \widetilde{V}^{ij} + K\left[-\sum_{j \in \mathcal{N}_i} w_{ij}\left(e_v^i - e_v^j\right) - h_{\overline{\sigma}}^i m_i e_v^i \right]. \end{aligned}$$

By the definition of function J, we have

$$\partial J = \left[\sum_{j=2}^{N} \left(\nabla_{e_p^1} \widetilde{V}^{1j} \right)^T, \ldots, \sum_{j=1}^{N-1} \left(\nabla_{e_p^N} \widetilde{V}^{Nj} \right)^T, m_1 e_v^{1T}, \ldots, m_N e_v^{NT} \right]^T,$$

and $\dot{J} \in^{a.e.} \dot{\widetilde{J}}$, where

$$\begin{aligned} \dot{\widetilde{J}} &\subseteq \sum_{i=1}^{N} \left(\nabla_{e_p^i} \widetilde{V}^i \right)^T e_v^i - e_v^T \begin{pmatrix} \vdots \\ \nabla_{e_p^i} \widetilde{V}^i \\ \vdots \end{pmatrix} \\ &\quad + e_v^T K\left[-(L_{\overline{\sigma}} \otimes I_n) e_v - (H_{\overline{\sigma}} \otimes I_n) e_v \right] \\ &= K\left[-e_v^T [(L_{\overline{\sigma}} + H_{\overline{\sigma}}) \otimes I_n] e_v \right], \end{aligned}$$

$\nabla_{e_p^i} \widetilde{V}^i = \sum_{j \in \mathcal{N}_i} \nabla_{e_p^i} \widetilde{V}^{ij}$ $e_v = (e_v^{1T}, \ldots, e_v^{NT})^T$ is the stack vector of all agent velocity vectors in the error system; $L_{\overline{\sigma}} = [l_{ij}]_{\overline{\sigma}} \in \mathbb{R}^{N \times N}$ with

$$l_{ij} = \begin{cases} -w_{ij}, & i \ne j, \\ \sum_{k=1, k \ne i}^{N} w_{ik}, & i = j; \end{cases}$$

$H_{\overline{\sigma}} = \mathrm{diag}(h_{\overline{\sigma}}^1 m_1, \ldots, h_{\overline{\sigma}}^N m_N)$; $\otimes$ stands for the Kronecker product; and I_n is the identity matrix of order n. Then, we get

$$\dot{\widetilde{J}} \subseteq -\overline{co}\left\{ e_v^T [(L_{\overline{\sigma}} + H_{\overline{\sigma}}) \otimes I_n] e_v \right\}.$$

It is easy to see that $L_{\overline{\sigma}}$ is symmetric and has the properties that each row sum is equal to 0, the diagonal entries are positive, and all the other entries are non-positive. Also, $H_{\overline{\sigma}}$ is a diagonal matrix with non-negative entries and there exists at least one positive diagonal entry. Furthermore, since the neighbouring graph $\mathcal{G}_{\overline{\sigma}}$ is connected, $L_{\overline{\sigma}} + H_{\overline{\sigma}}$ is irreducible. Hence, matrix $L_{\overline{\sigma}} + H_{\overline{\sigma}}$ is irreducibly diagonally dominant. By Corollary 6.2.27 in Horn and Johnson (1985), it follows that matrix $L_{\overline{\sigma}} + H_{\overline{\sigma}}$ is positive definite. Thus $-\overline{co}\{e_v^T [(L_{\overline{\sigma}} + H_{\overline{\sigma}}) \otimes I_n] e_v\}$ is an interval of the form $[l, 0]$ with $l < 0$. Therefore, for any $z \in \dot{\widetilde{J}}$, $z \le 0$, and only when $e_v^1 = \cdots = e_v^N = \mathbf{0}$, 0 is contained in $-\overline{co}\{e_v^T [(L_{\overline{\sigma}} + H_{\overline{\sigma}}) \otimes I_n] e_v\}$. This occurs only when $v^1 = \cdots = v^N = v^0$. It follows that $\dot{e}_v^i = \mathbf{0}$, $i = 1, \ldots, N$, and $\dot{e}_p^{ij} = \mathbf{0}$, $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$. Hence, according to non-smooth version of LaSalle's invariance principle (Shevitz and Paden 1994), we know that the Filippov solution trajectories of the system converge to the largest invariant subset of the set defined by $S = \{e_v \in \Omega \mid 0 \in \dot{\widetilde{J}}\}$. In the set, the agent velocity dynamics is $\dot{e}_v^i = -(1/m_i) \sum_{j \in \mathcal{N}_i} \nabla_{e_p^i} \widetilde{V}^{ij} = -(1/m_i) \nabla_{e_p^i} \widetilde{V}^i$. Thus, in steady state, all agent velocities in the error system no longer change and equal zero, and moreover, the potential $\widetilde{V}^i$ of each agent i is minimised. Freedom from collisions between the agents can be ensured since otherwise it will result in $\widetilde{V}^i \to +\infty$. □

Remark 1: Note that in case all agents know the desired velocity and all coefficients $h_i, i = 1, \ldots, N$, are used to design the control law, the connectivity condition in Theorems 1 and 2 can be weakened. That is, all the agents can still move at the desired velocity eventually even if the neighbouring graph is not connected. In this case, the term α_i in the control law (4) should be remained to make sure collision-free in the group. However, it is difficult to give a theoretical analysis. Instead, we will present some simulation results in §4 to demonstrate this case.

Remark 2: By the analysis above, it is easy to see that, when the coefficient h_s^i equals h_i for all $i \in \mathcal{I}$ in the control law (4), the strong flocking motion can still be obtained though all agents do not regulate their velocities according to their neighbouring agents, i.e., we can omit the term β^i in (2).

Remark 3: If the coupling matrix W_σ is asymmetric, we can regulate the control law acting on each agent to generate the desired strong flocking motion. The main analysis is as follows.

Define the position neighbouring graph $\mathcal{G}_\sigma$ and the velocity neighbouring graph $\mathcal{D}_{\sigma^*}$ as in Shi et al. (2006a) and assume that $\mathcal{G}_\sigma$ and $\mathcal{D}_{\sigma^*}$ are always strongly connected, where σ^* is a switching signal and is a piecewise constant function $\sigma^*(t)\colon [0,+\infty) \to \mathcal{P}^*$, $\mathcal{P}^*$ is a finite index set where the number of the indices is equal to the number of the strongly connected graph $\mathcal{D}_{\sigma^*}$ in the group. From Shi et al. (2006b), we obtain that, if $\mathcal{D}_{\sigma^*}$ is strongly connected, then its Laplacian matrix L_{σ^*} is irreducible and for each L_{σ^*}, there is only one left eigenvector $\xi_{\sigma^*} = [\xi_1, \ldots, \xi_N]^T_{\sigma^*} \in \mathbb{R}^N$ such that $0 < \xi_i \le 1$ for all $i \in \mathcal{I}$, $\xi^T_{\sigma^*} L_{\sigma^*} = 0$, and $\sum_{i=1}^N \xi_i = 1$. Then, we modify the control law u^i to

$$u^i = \underbrace{-\sum_{j\in\mathcal{N}_i} \nabla_{x^i} V^{ij}}_{\alpha^i} \underbrace{-\sum_{j\in\mathcal{N}_i^*} \xi_i w_{ij}\big(v^i - v^j\big)}_{\beta^i} \underbrace{-h_s^i m_i\big(v^i - v^0(t)\big) + g_i m_i a_0(t)}_{\gamma^i} + \underbrace{k_i v^i}_{\delta^i},$$

where $\mathcal{N}_i^* \triangleq \{ j \mid w_{ij} > 0\}$. By a similar analysis, we get

$$\dot{\tilde{J}} \subseteq -\frac{1}{2}\overline{co}\big\{e_v^T\big[\big(\Lambda_{\overline{\sigma}} L_{\overline{\sigma}} + L_{\overline{\sigma}}^T \Lambda_{\overline{\sigma}} + 2H_{\overline{\sigma}}\big) \otimes I_n\big] e_v\big\},$$

where $\Lambda_{\overline{\sigma}} = \operatorname{diag}(\xi_1, \ldots, \xi_N)_{\overline{\sigma}} \in \mathbb{R}^{N\times N}$, where $\overline{\sigma}$ is a switching signal and is a piecewise constant function $\overline{\sigma}(t)\colon [0, +\infty) \to \overline{\mathcal{P}}$, $\overline{\mathcal{P}}$ is a finite index set where the number of the indices is equal to the number of the elements of set $\{(x,y,z) \mid \forall x \in \mathcal{P}, \forall y \in \mathcal{P}^*, \forall z \in \widehat{\mathcal{P}}\}$. It is easy to see that $\Lambda_{\overline{\sigma}} L_{\overline{\sigma}} + L_{\overline{\sigma}}^T \Lambda_{\overline{\sigma}}$ is symmetric and has the properties that each row sum is equal to 0, the diagonal entries are positive, and all the other entries are non-positive. The rest analysis is similar to Theorem 2, and thus is omitted.

3.1.2 *The motion of the CoM*

In what follows, we will analyse the motion of system (5) in the case where $h_s^i = h_0$ for all $i \in \mathcal{I}$, where $h_0 > 0$ is a constant. This means that the intensities of influence of the external signal on all agents are equal. Hence, the control law in (4) has the following form:

$$u^i = \underbrace{-\sum_{j\in\mathcal{N}_i} \nabla_{x^i} V^{ij}}_{\alpha^i} \underbrace{-\sum_{j\in\mathcal{N}_i} w_{ij}\big(v^i - v^j\big)}_{\beta^i} \underbrace{-h_0 m_i\big(v^i - v^0(t)\big) + m_i a_0(t)}_{\gamma^i} + \underbrace{k_i v^i}_{\delta^i}. \qquad (8)$$

Certainly, in this case, we also have the conclusions in Theorems 1 and 2. Here we will choose another moving reference frame to study the flocking problem.

The position vector of the CoM in system (5) is defined as

$$x^* = \frac{\sum_{i=1}^N m_i x^i}{\sum_{i=1}^N m_i}.$$

Thus, the velocity vector of the CoM is

$$v^* = \frac{\sum_{i=1}^N m_i v^i}{\sum_{i=1}^N m_i}.$$

On using control law (8) and by the symmetry of W_σ and the symmetry of function V^{ij} with respect to x^{ij}, we get

$$\dot{v}^* = -h_0 v^* + h_0 v^0(t) + a_0(t). \qquad (9)$$

By solving (9), we get

$$v^*(t) = v^0(t) + \big(v^*(t_0) - v^0(t_0)\big) e^{-h_0(t-t_0)}.$$

Thus, it follows that, if $v^*(t_0) = v^0(t_0)$, then the velocity of the CoM equals $v^0(t)$ for all time; if $v^*(t_0) \neq v^0(t_0)$, then the velocity of the CoM exponentially converges to the desired velocity $v^0(t)$ with a time constant of h_0 s. Moreover, since $\dot{x}^* = v^*$, we have

$$x^*(t) = x^*(t_0) + \int_{t_0}^t v^0(\tau)\mathrm{d}\tau + \frac{v^*(t_0) - v^0(t_0)}{h_0}\big[1 - e^{-h_0(t-t_0)}\big].$$

We define the error vectors:

$$e_p^* = x^* - \int_{t_0}^t v^0(\tau)\mathrm{d}\tau \quad \text{and} \quad e_v^* = v^* - v^0(t).$$

Then e_p^* represents the position difference vector between the CoM and the virtual leader, whereas e_v^* represents the velocity difference vector between them. By the calculation above, it is easy to see that

$$\lim_{t\to+\infty} e_p^* = x^*(t_0) + \frac{v^*(t_0) - v^0(t_0)}{h_0}.$$

Thus, it follows that, if $v^*(t_0) = v^0(t_0)$, then the position difference between the CoM and the virtual leader equals $x^*(t_0)$ for all time; if $v^*(t_0) \neq v^0(t_0)$, then the difference exponentially approaches the constant vector $x^*(t_0) + (v^*(t_0) - v^0(t_0))h_0$ with a time constant of h_0 s. Therefore, from the analysis above, we have the following theorem.

Theorem 3: *Taking the control law in* (8), *if the initial velocity of the CoM is equal to the desired initial* *velocity, then the velocity of the CoM equals the desired velocity for all time and the position difference between the CoM and the virtual leader always equals $x^*(t_0)$; otherwise the velocity of the CoM will exponentially converge to the desired velocity with a time constant of h_0 s and the position difference between the CoM and the virtual leader will exponentially approach the constant vector $x^*(t_0)+[(v^*(t_0)-v^0(t_0))/h_0]$.*

In this case, we can also choose the moving reference frame proposed in Olfati-Saber (2006) to analyse the stability of system, and take the position of the CoM of the group as the origin (called 'the second moving reference frame'). We define the error vectors

$$\varepsilon_p^i = x^i - x^* \quad \text{and} \quad \varepsilon_v^i = v^i - v^*.$$

Then ε_v^i represents the velocity difference vector between agent i and the CoM. It is easy to see that $\dot{\varepsilon}_p^i = \varepsilon_v^i$ and $\dot{\varepsilon}_v^i = \dot{v}^i - \dot{v}^*$. Hence, the error dynamics is given by

$$\left.\begin{aligned} \dot{\varepsilon}_p^i &= \varepsilon_v^i, \\ m_i\dot{\varepsilon}_v^i &\in^{a.e.} K\big[u^i - k_i v^i - m_i\dot{v}^*\big], \quad i=1,\ldots,N. \end{aligned}\right\} \tag{10}$$

By the definition of V^{ij}, it follows that $V^{ij}(\|x^{ij}\|) = V^{ij}(\|\varepsilon_p^{ij}\|) \triangleq \overline{V}^{ij}$, where $\varepsilon_p^{ij} = \varepsilon_p^i - \varepsilon_p^j$ and hence $\overline{V}^i = V^i$ and $\nabla_{\varepsilon_p^i}\overline{V}^{ij} = \nabla_{x^i}V^{ij}$. Thus the control input u^i for agent i in the error system (10) has the following form:

$$u^i = \underbrace{-\sum_{j\in\mathcal{N}_i}\nabla_{\varepsilon_p^i}\overline{V}^{ij}}_{\alpha^i}\underbrace{-\sum_{j\in\mathcal{N}_i}w_{ij}\big(\varepsilon_v^i-\varepsilon_v^j\big)}_{\beta^i}$$
$$\underbrace{-h_0m_i\varepsilon_v^i - h_0m_ie_v^* + m_ia_0(t)}_{\gamma^i} + \underbrace{k_iv^i}_{\delta^i}.$$

We consider the error system (10) and choose the following Lyapunov function:

$$J = \frac{1}{2}\sum_{i=1}^{N}\Big(\overline{V}^i + m_i\varepsilon_v^{iT}\varepsilon_v^i\Big). \tag{11}$$

By a similar calculation, we get $\dot{\widetilde{J}} \subseteq -\overline{co}\{\varepsilon_v^T \times [(L_\sigma + H_0)\otimes I_n]\varepsilon_v\}$, where $\varepsilon_v = (\varepsilon_v^{1T},\ldots,\varepsilon_v^{NT})^T$ and $H_0 = \text{diag}(h_0m_1,\ldots,h_0m_N)\in\mathbb{R}^{N\times N}$. Using the analysis method in Theorem 2, we obtain that the velocities of all agents asymptotically approach the velocity of the CoM, avoidance of collisions between the agents is ensured, and the group final configuration minimises all agent potentials. Furthermore, from Theorem 3, we conclude that the velocities of all agents in group (5) asymptotically approach the desired velocity.

Remark 4: One issue to be mentioned here is that, when the intensities of influence of the external signal on the motions of all agents are not equal, it is difficult to estimate the motion of the CoM and analyse the stability properties of system (5) under the second moving reference frame.

3.2 *Flocking control: group motion with common but unknown velocity*

In this case, we assume that all agents know the desired velocity $v^0(t)$, but none of them know the acceleration $a_0(t)$. Moreover, we still assume that the coefficients $h_s^i = h_0 > 0$ and $g_i = 0$ for all $i\in\mathcal{I}$. Thus the control law acting on agent i is

$$u^i = \underbrace{-\sum_{j\in\mathcal{N}_i}\nabla_{x^i}V^{ij}}_{\alpha^i}\underbrace{-\sum_{j\in\mathcal{N}_i}w_{ij}\big(v^i-v^j\big)}_{\beta^i}$$
$$\underbrace{-h_0m_i\big(v^i - v^0(t)\big)}_{\gamma^i} + \underbrace{k_iv^i}_{\delta^i}, \tag{12}$$

where w_{ij}, $v^0(t)$, and h_0 are defined as before. In what follows, we will show that all agents cannot move ahead at the desired velocity, but in this case, the weak flocking motion of the agent group can be achieved.

Theorem 4: *Taking the control law in* (12), *all agent velocities in the group described in* (5) *become asymptotically the same, avoidance of collisions between the agents is ensured, and the group final configuration minimises all agent potentials.*

This theorem is true after Theorem 5 is proved. First, on using control law (12), we get

$$\dot{v}^* = -h_0v^* + h_0v^0(t). \tag{13}$$

We consider the error dynamics (10). The control input u^i for agent i in the error system has the following form:

$$u^i = \underbrace{-\sum_{j\in\mathcal{N}_i}\nabla_{\varepsilon_p^i}\overline{V}^{ij}}_{\alpha^i}\underbrace{-\sum_{j\in\mathcal{N}_i}w_{ij}\big(\varepsilon_v^i-\varepsilon_v^j\big)}_{\beta^i}$$
$$\underbrace{-h_0m_i\varepsilon_v^i - h_0m_ie_v^*}_{\gamma^i} + \underbrace{k_iv^i}_{\delta^i}. \tag{14}$$

Theorem 5: *Taking the control law in* (14), *all agent velocities in the error system* (10) *asymptotically approach zero, avoidance of collisions between the agents is ensured, and the group final configuration minimises all agent potentials.*

Choosing the Lyapunov function J defined as in (11) and calculating the derivative of J along the solution of the error system (10), we have $\dot{J} \in^{a.e.} \dot{\widetilde{J}} \subseteq -\overline{co}\{\varepsilon_v^T[(L_\sigma + H_0)\otimes I_n]\varepsilon_v\}$. Following the analysis method in Theorem 2, we can obtain the proof of Theorem 5. Here we omit the detailed proof.

Theorem 5 implies that all agent velocities in group (5) asymptotically approach the velocity of the CoM by using control law (12), but in what follows, we will show that in this case the final velocity of the group may not asymptotically approach the desired velocity. In fact, for some cases, all agents can track the external signal, but for others, they cannot. We will present two simple examples, which is enough to illustrate the problem. By solving (13), we have

$$v^*(t) = v^*(t_0)e^{-h_0(t-t_0)} + h_0\int_{t_0}^{t} e^{-h_0(t-\tau)}v^0(\tau)\mathrm{d}\tau,$$

and moreover, we obtain that

$$x^*(t) = x^*(t_0) + \frac{v^*(t_0)}{h_0}\left[1 - e^{-h_0(t-t_0)}\right] + h_0\int_{t_0}^{t}\int_{t_0}^{s} e^{-h_0(s-\tau)}v^0(\tau)\mathrm{d}\tau\,\mathrm{d}s.$$

Example 1: Suppose the desired velocity $v^0(t)$ is a constant vector v_0, then we get

$$v^*(t) = v_0 + (v^*(t_0) - v_0)e^{-h_0(t-t_0)}.$$

It is obvious that the velocity of the CoM equals the desired velocity for all time or it will exponentially converge to it with a time constant of h_0 s. Furthermore, by Theorem 5, we obtain that the velocities of all agents asymptotically approach the desired velocity. Moreover, we have

$$x^*(t) = x^*(t_0) + v_0(t - t_0) + \frac{v^*(t_0) - v_0}{h_0}\left[1 - e^{-h_0(t-t_0)}\right],$$

hence,

$$\lim_{t\to+\infty} e_p^* = x^*(t_0) + \frac{v^*(t_0) - v_0}{h_0}.$$

This implies that the position difference between the CoM and the virtual leader will asymptotically approach a constant vector. By the analysis above, we know that, when the desired velocity is a constant vector, the desired stable flocking motion can be obtained by using control law (12). More information can be found in Shi et al. (2005) and Shi et al. (2006a).

Example 2: Suppose $n=1$ and $v^0(t)=\alpha t$, where α is a positive constant, then we get

$$v^*(t) = \alpha t + (v^*(t_0) - \alpha t_0)e^{-h_0(t-t_0)} - \frac{\alpha}{h_0}\left[1 - e^{-h_0(t-t_0)}\right].$$

It is easy to see that $\lim_{t\to+\infty} e_v^* = -(\alpha/h_0)$. Moreover, we have

$$x^*(t) = x^*(t_0) + \frac{\alpha}{2}\left(t^2 - t_0^2\right) - \frac{\alpha}{h_0}(t - t_0) + \left(\frac{v^*(t_0) - \alpha t_0}{h_0} + \frac{\alpha}{h_0^2}\right)\left[1 - e^{-h_0(t-t_0)}\right],$$

thus $\lim_{t\to+\infty} e_p^* = -\infty$. This implies that the velocity of the CoM cannot asymptotically approach the desired velocity, i.e., the CoM cannot track the external signal. Hence, in this case, the desired stable flocking motion cannot be obtained by using the control law in (12).

In what follows, we will demonstrate that in this case the strong flocking motion still may not be achieved even when the position information of the virtual leader is considered in the design of the control law. The initial position of the virtual leader is still chosen as the origin, then its position vector is $x^0(t) = \int_{t_0}^{t} v^0(\tau)\mathrm{d}\tau$. We modify the control law u^i in (12) to

$$u^i = \underbrace{-\sum\nolimits_{j\in\mathcal{N}_i}\nabla_{x^i}V^{ij}}_{\alpha^i}\underbrace{-\sum\nolimits_{j\in\mathcal{N}_i}w_{ij}\left(v^i - v^j\right)}_{\beta^i} \underbrace{-h_0 m_i\left(v^i - v^0(t)\right) - r_0 m_i\left(x^i - x^0(t)\right)}_{\gamma^i} + \underbrace{k_i v^i}_{\delta^i}, \tag{15}$$

where $r_0 > 0$ is a constant. By a similar calculation, we get

$$\dot{v}^* = -h_0 v^* + h_0 v^0(t) - r_0 x^* + r_0 x^0(t).$$

We consider the error system (10) and choose the following Lyapunov function:

$$J = \frac{1}{2}\sum_{i=1}^{N}\left(\overline{V}^i + m_i\varepsilon_v^{iT}\varepsilon_v^i + r_0 m_i\varepsilon_p^{iT}\varepsilon_p^i\right).$$

Then we have $\dot{\tilde{J}} \subseteq -\overline{co}\{\varepsilon_v^T[(L_\sigma + H_0)\otimes I_n]\varepsilon_v\}$. The rest of the analysis is similar, and thus is omitted. Hence, on using the control law in (15), the velocities of all agents in group (5) still asymptotically approach the velocity of the CoM.

Next, we analyse the motion of the CoM. By the calculation above, we have

$$\begin{bmatrix}\dot{x}^*\\ \dot{v}^*\end{bmatrix} = \begin{bmatrix}0 & I_n\\ -r_0 I_n & -h_0 I_n\end{bmatrix}\begin{bmatrix}x^*\\ v^*\end{bmatrix} + \begin{bmatrix}0\\ h_0 v^0 + r_0 x^0\end{bmatrix}, \tag{16}$$

where $v^0 \triangleq v^0(t)$ and $x^0 \triangleq x^0(t)$. Let

$$A = \begin{bmatrix}0 & I_n\\ -r_0 I_n & -h_0 I_n\end{bmatrix} = \begin{bmatrix}0 & 1\\ -r_0 & -h_0\end{bmatrix}\otimes I_n.$$

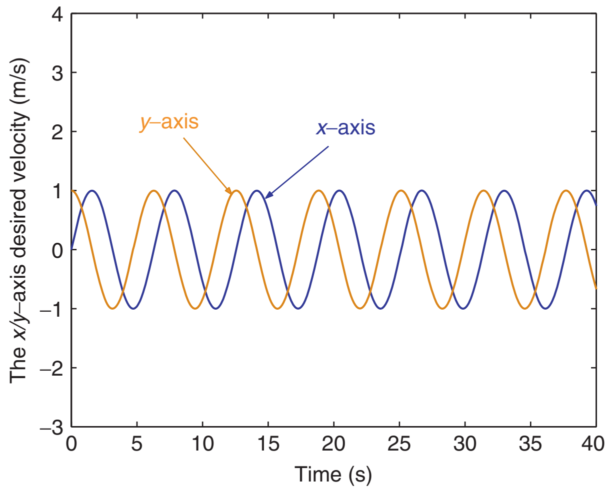

Figure 2. The desired velocity curves.

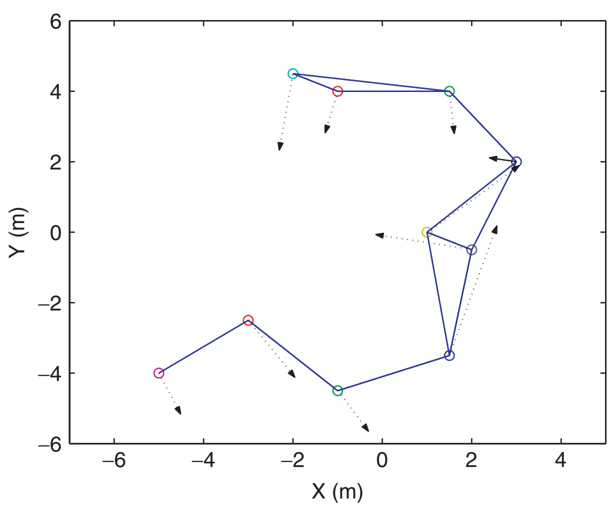

Figure 3. The initial state of the agent group.

In the following, we will present an example to illustrate the fact that the velocity of the CoM may not asymptotically approach the desired value $v^0(t)$ by using control law (15).

Example 3: Suppose $t_0=0$, $n=1$, and $v^0(t)=e^{\alpha h_0 t}$, then $x^0(t)=(1/\alpha h_0)(e^{\alpha h_0 t}-1)$, where α is a positive constant. Thus the eigenvalues of matrix A is

$$\lambda_1=\frac{-h_0+\sqrt{h_0^2-4r_0}}{2} \quad \text{and} \quad \lambda_2=\frac{-h_0-\sqrt{h_0^2-4r_0}}{2},$$

and they all are negative or have the negative real parts.

(i) If $h_0^2-4r_0\neq 0$, then $\lambda_1\neq\lambda_2$ and the eigenvectors associated with them are $(1,\lambda_1)^T$ and $(1,\lambda_2)^T$, respectively. Let $P=\left[\begin{smallmatrix}1 & 1\\ \lambda_1 & \lambda_2\end{smallmatrix}\right]$, then $P^{-1}=1/(\lambda_2-\lambda_1)\left[\begin{smallmatrix}\lambda_2 & -1\\ -\lambda_1 & 1\end{smallmatrix}\right]$. Thus $A=P\mathrm{diag}(\lambda_1,\lambda_2)P^{-1}$. By solving (16), we obtain

$$\begin{bmatrix} x^* \\ v^* \end{bmatrix}=\begin{bmatrix} \dfrac{\alpha h_0^2+r_0}{\alpha h_0[(\alpha^2+\alpha)h_0^2+r_0]}e^{\alpha h_0 t}-\dfrac{1}{\alpha h_0}+(*)_1 \\ \dfrac{\alpha h_0^2+r_0}{(\alpha^2+\alpha)h_0^2+r_0}e^{\alpha h_0 t}+(*)_2 \end{bmatrix},$$

where $\lim_{t\to+\infty}(*)_1=0$ and $\lim_{t\to+\infty}(*)_2=0$. Hence, we get

$$\lim_{t\to+\infty} e_v^* = \lim_{t\to+\infty}\left[\frac{-\alpha^2 h_0^2}{(\alpha^2+\alpha)h_0^2+r_0}e^{\alpha h_0 t}+(*)_2\right]=-\infty,$$

and

$$\lim_{t\to+\infty} e_p^* = \lim_{t\to+\infty}\left[\frac{-\alpha h_0}{(\alpha^2+\alpha)h_0^2+r_0}e^{\alpha h_0 t}+(*)_1\right]=-\infty.$$

(ii) If $h_0^2-4r_0=0$, then $\lambda_1=\lambda_2=-(h_0/2)$ and the eigenvectors associated with them are $(1,\lambda_1)^T$ and $(1,1+\lambda_1)^T$. Let

$$P=\begin{bmatrix}1 & 1\\ \lambda_1 & 1+\lambda_1\end{bmatrix} \quad \text{and} \quad J_0=\begin{bmatrix}\lambda_1 & 1\\ 0 & \lambda_1\end{bmatrix},$$

then

$$P^{-1}=\begin{bmatrix}1+\lambda_1 & -1\\ -\lambda_1 & 1\end{bmatrix} \quad \text{and} \quad A=PJ_0P^{-1}.$$

By solving (16), we have

$$\begin{bmatrix} x^* \\ v^* \end{bmatrix}=\begin{bmatrix} \dfrac{4\alpha+1}{\alpha h_0(2\alpha+1)^2}e^{\alpha h_0 t}-\dfrac{1}{\alpha h_0}+(*)_3 \\ \dfrac{4\alpha+1}{(2\alpha+1)^2}e^{\alpha h_0 t}+(*)_4 \end{bmatrix},$$

where $\lim_{t\to+\infty}(*)_3=0$ and $\lim_{t\to+\infty}(*)_4=0$. Hence, we get

$$\lim_{t\to+\infty} e_v^* = \lim_{t\to+\infty}\left[\frac{-4\alpha^2}{(2\alpha+1)^2}e^{\alpha h_0 t}+(*)_4\right]=-\infty,$$

and

$$\lim_{t\to+\infty} e_p^* = \lim_{t\to+\infty}\left[\frac{-4\alpha}{h_0(2\alpha+1)^2}e^{\alpha h_0 t}+(*)_3\right]=-\infty.$$

From the analysis above, we conclude that the CoM cannot track the virtual leader and thus the strong flocking cannot be achieved by using the control law in (15).

Remark 5: Note that, when none of the agents know the acceleration $a_0(t)$, it is difficult to analyse the stability properties of system (5) under the first moving reference frame.

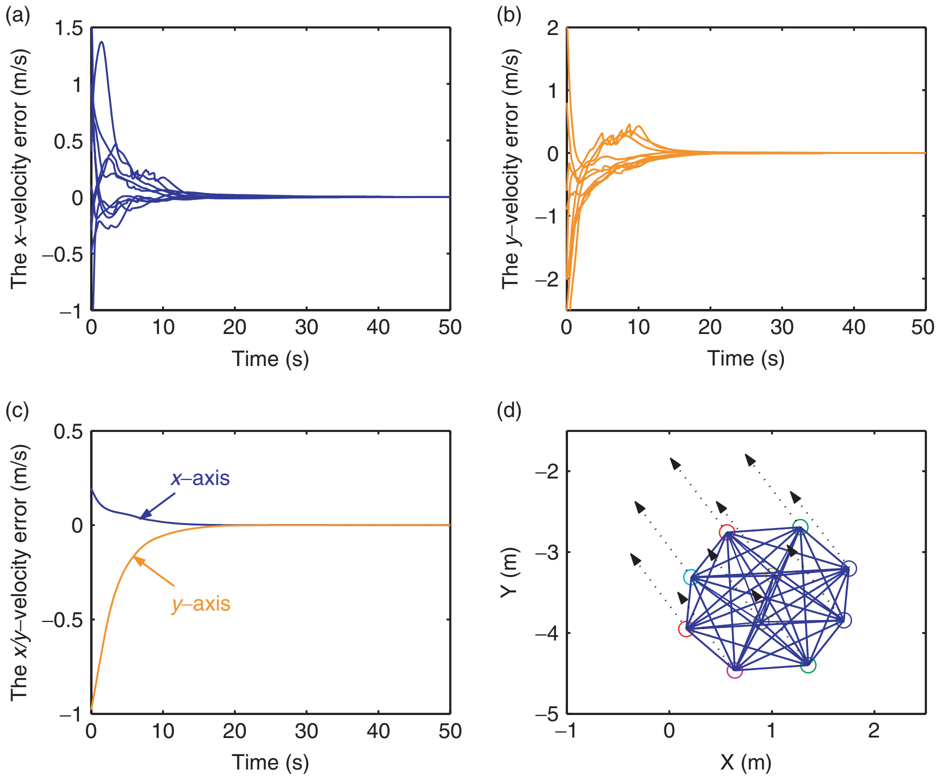

Figure 4. (a) and (b) depict the curves of the velocity errors between the agents and the desired velocity along x-axis and y-axis, respectively, and (c) plots the velocity error between the CoM and the desired velocity. (d) presents the final group configuration and all agents' velocities at $t = 100$ s. (Here $h_s^i = h_i$ for some $i \in \mathcal{I}_0$ and $g_i = 1$ for all $i \in \mathcal{I}_0$.)

Remark 6: It should be mentioned here that, in the case where some agents know the acceleration $a_0(t)$ but others do not, the flocking motion cannot be achieved. This will be demonstrated by the simulations in the next section.

4. Numerical simulations

In this section, we will present some numerical simulations for system (5) in order to illustrate the theoretical results obtained in the previous sections.

These simulations are performed with ten agents, labelled with circles, moving in the plane, whose initial positions, velocities and neighbouring relations are set randomly, but which satisfy: (1) all initial positions are set within a circle of radius of $R^* = 10$ m centred at the origin; (2) all initial velocities are set with arbitrary directions and magnitudes within the range of [0, 4] m/s; and (3) the initial neighbouring graph is connected. All agents have different masses and they are set randomly in the range of (0, 1] kg. Suppose the desired velocity $v^0(t) = [\sin(t), \cos(t)]^T$ and the initial time $t_0 = 0$ s. We run the simulations for 100 s and choose suitable coordinate axes to show our simulation results.

Figure 2 depicts the curves of the desired velocity along x-axis and y-axis. Figures 4–12 show the simulation results for the same group, and the group has the same initial state shown in Figure 3 where the solid lines represent the neighbouring relations between agents and the dotted arrows represent the initial velocities of all agents. However, different control laws are taken in the form of (4) (in Figures 4, 5, 6, 10 and 11), (12) (in Figure 7), or (15) (in Figure 8) with the explicit potential function (3), where $a = b = 0.05$. The agent's sensing range is chosen as $R = 4$ m. In Figures 4–6 and 10–11, $h_s^i = h_i$ or $h_s^i = 0$ where h_i is generated randomly such that $0 < hi < 1$, $\forall i \in \mathcal{I}_0 = \{1, \dots, 10\}$, and for each $i \in \mathcal{I}_0$, h_i may be different in different simulations. Take $h_0 = r_0 = 1$ in Figures 7 and 8. The interaction coefficient w_{ij} equals c_{ij} if agent j is a neighbour of agent i and is 0 otherwise, where the coefficient c_{ij} is generated randomly such that $0 < c_{ij} = c_{ji} < 1$ and $c_{ii} = 0$ for all $i, j = 1, \dots, 10$.

Figures 4–6 present the simulation results in the case where all agents know the desired velocity and the acceleration, and the coefficients h_i, $i = 1, \dots, 10$, are not equal to each other. In Figure 4, the coefficient h_s^i equals h_i for some $i \in \mathcal{I}_0$ in control law (4), whereas in Figure 5, the coefficient h_s^i equals h_i for all $i \in \mathcal{I}_0$.

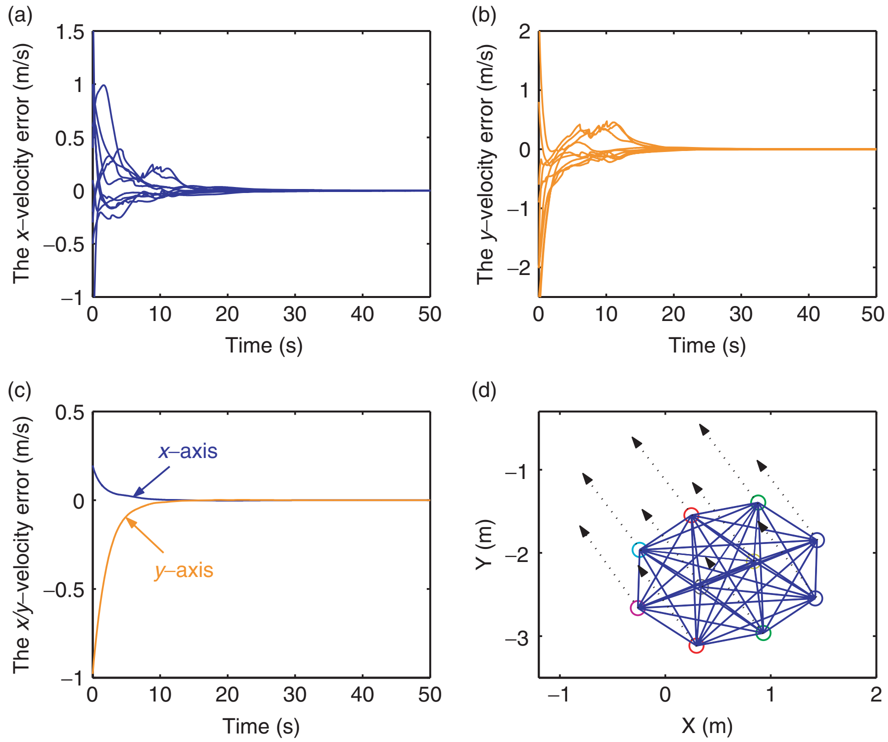

Figure 5. (a) and (b) depict the curves of the velocity errors between the agents and the desired velocity along x-axis and y-axis, respectively, and (c) plots the velocity error between the CoM and the desired velocity. (d) presents the final group configuration and all agents' velocities at $t = 100$ s. (Here $h_s^i = h_i$ and $g_i = 1$ for all $i \in \mathcal{I}_0$.)

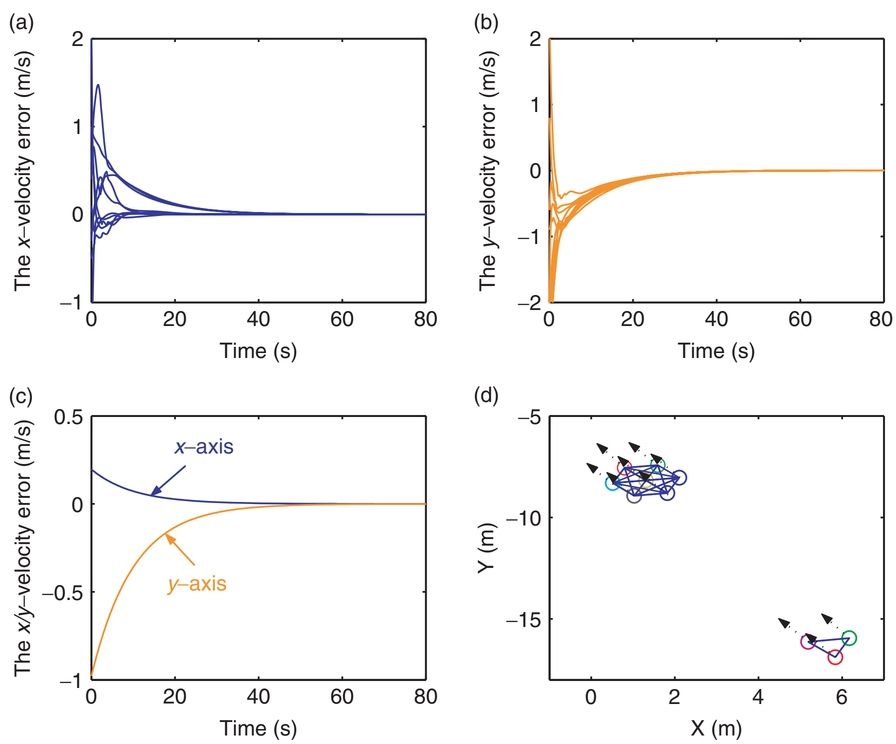

Figure 6. (a) and (b) depict the curves of the velocity errors between the agents and the desired velocity along x-axis and y-axis, respectively, and (c) plots the velocity error between the CoM and the desired velocity. (d) presents the final group configuration and all agents' velocities at $t = 100$ s. (Here $h_s^i = h_i$ and $g_i = 1$ for all $i \in \mathcal{I}_0$.)

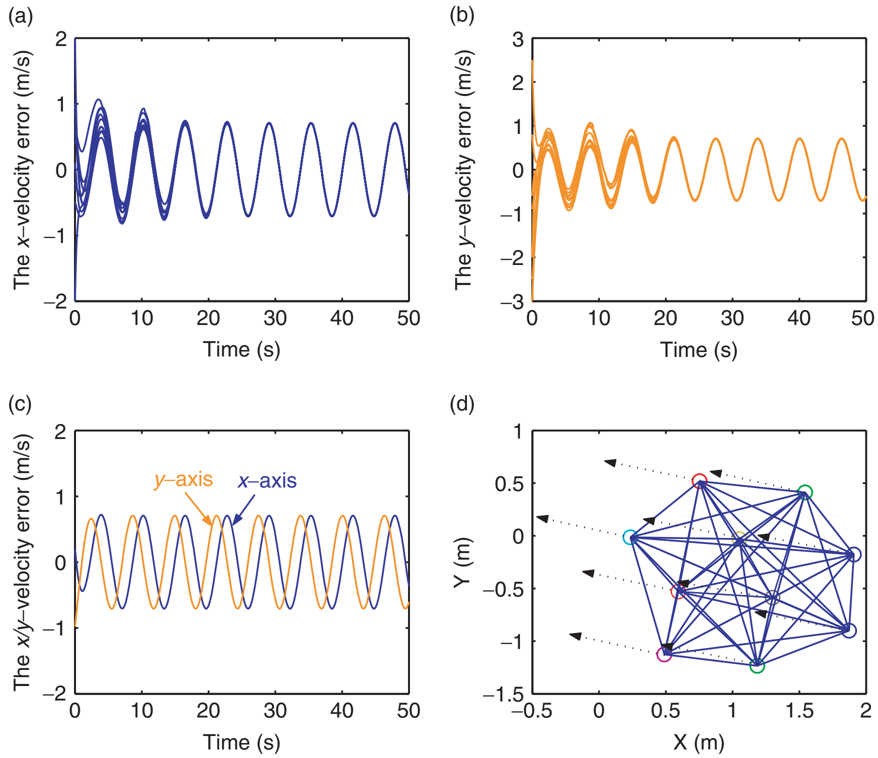

Figure 7. (a) and (b) depict the curves of the velocity errors between the agents and the desired velocity along x-axis and y-axis, respectively, and (c) plots the velocity error between the CoM and the desired velocity. (d) presents the final group configuration and all agents' velocities at $t = 100$ s. (Here $h_0 = 1$ and $g_i = 0$ for all $i \in \mathcal{I}_0$.)

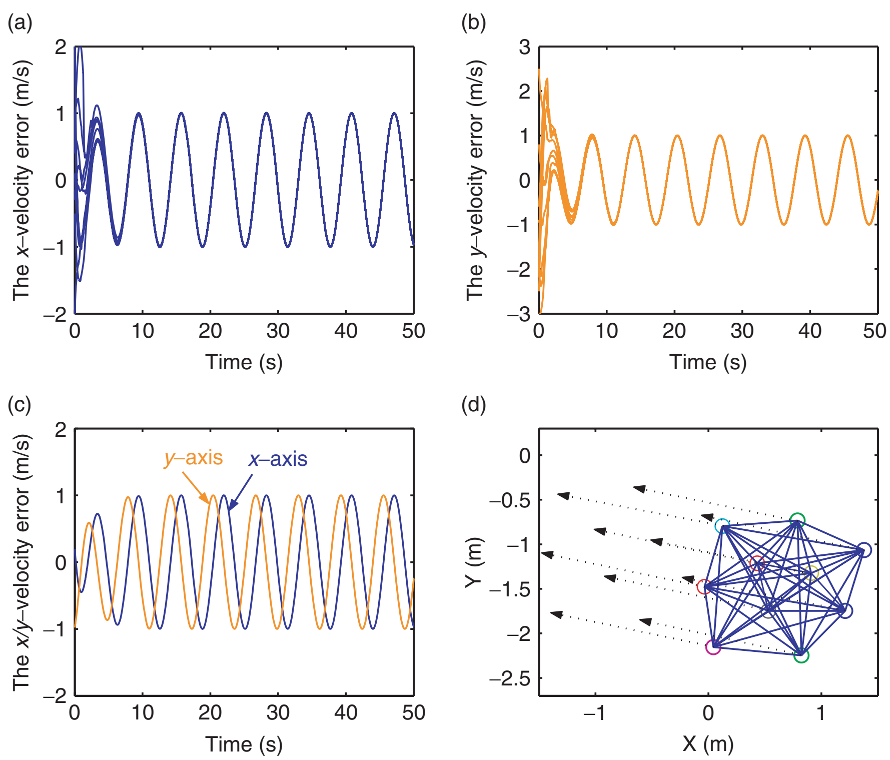

Figure 8. (a) and (b) depict the curves of the velocity errors between the agents and the desired velocity along x-axis and y-axis, respectively, and (c) plots the velocity error between the CoM and the desired velocity. (d) presents the final group configuration and all agents' velocities at $t = 100$ s. (Here $h_0 = r_0 = 1$ and $g_i = 0$ for all $i \in \mathcal{I}_0$.)

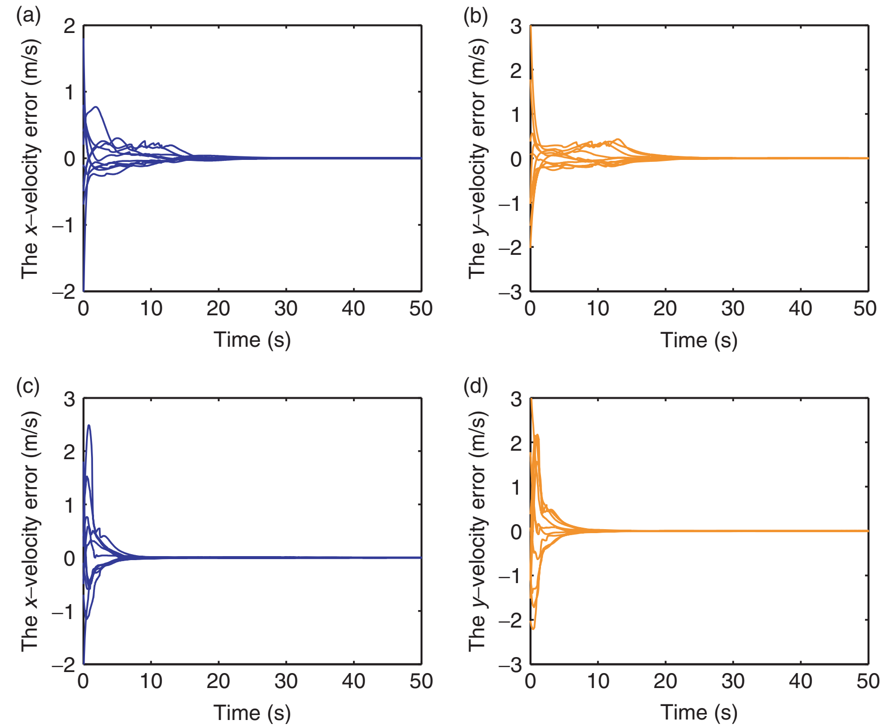

Figure 9. (a)–(b) and (c)–(d) depict the curves of the velocity errors between the agents and the CoM along x-axis and y-axis in the simulations shown in Figures 7 and 8, respectively.

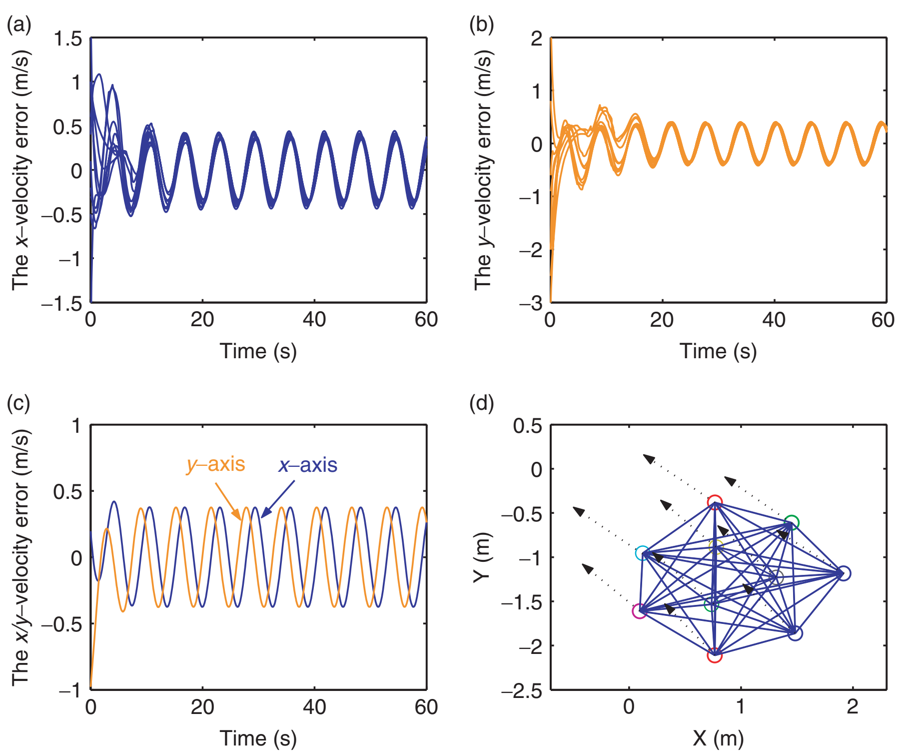

Figure 10. (a) and (b) depict the curves of the velocity errors between the agents and the desired velocity along x-axis and y-axis, respectively, and (c) plots the velocity error between the CoM and the desired velocity. (d) presents the group configuration and all agents' velocities at $t=100$ s. (Here $h_s^i=h_i$ for all $i\in\mathcal{I}_0$ and $g_i=1$ for some $i\in\mathcal{I}_0$.)

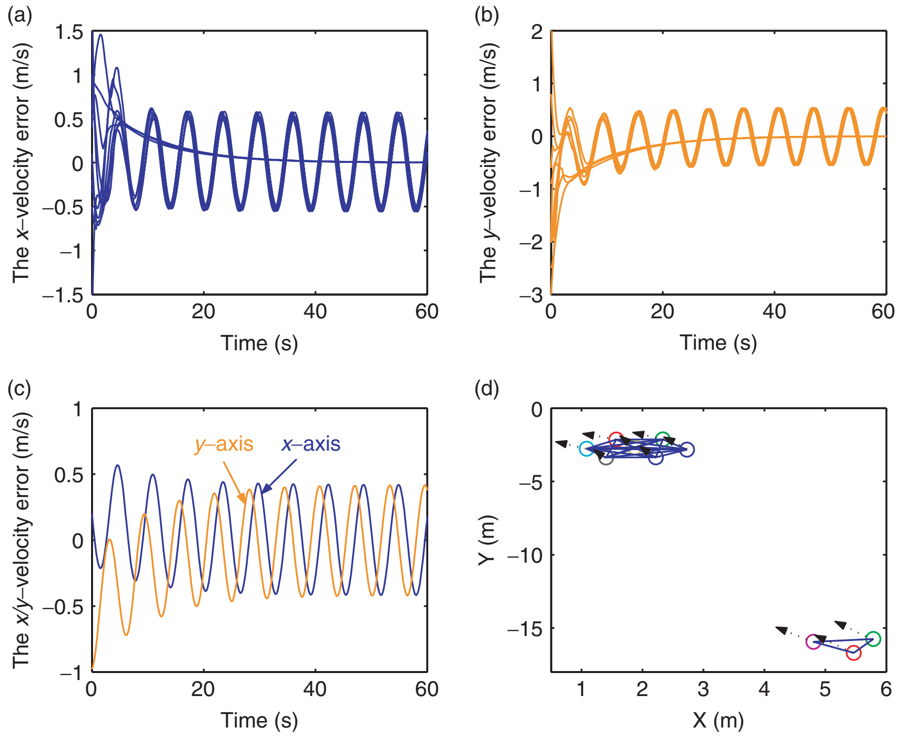

Figure 11. (a) and (b) depict the curves of the velocity errors between the agents and the desired velocity along x-axis and y-axis, respectively, and (c) plots the velocity error between the CoM and the desired velocity. (d) presents the group configuration and all agents' velocities at $t = 100$ s. (Here $h_s^i = h_i$ for all $i \in \mathcal{I}_0$ and $g_i = 1$ for some $i \in \mathcal{I}_0$.)

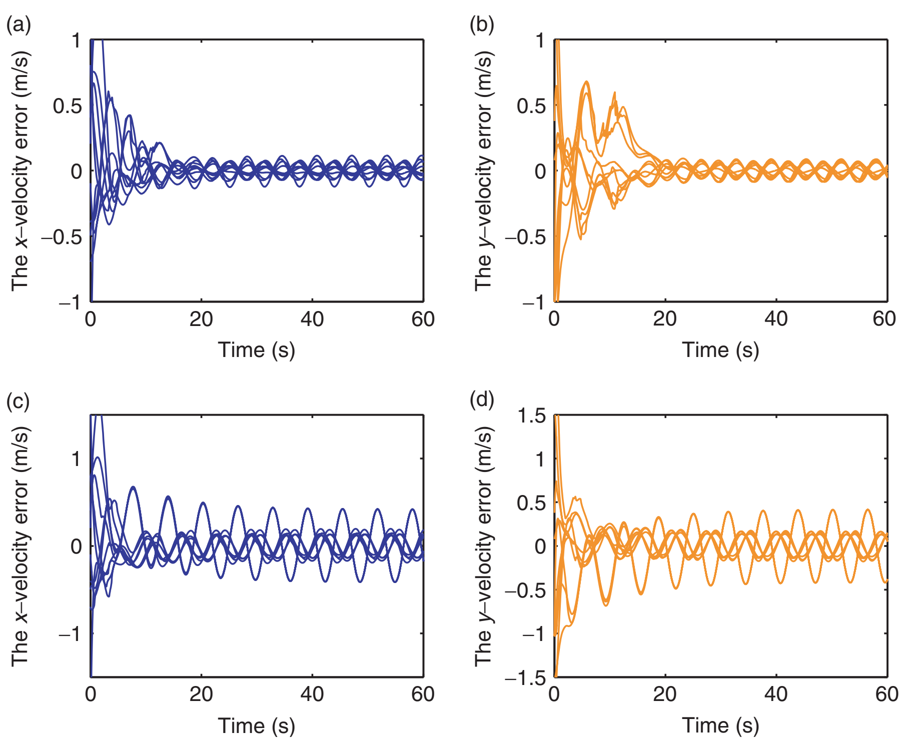

Figure 12. (a)–(b) and (c)–(d) depict the curves of the velocity errors between the agents and the CoM along x-axis and y-axis in the simulations shown in Figures 10 and 11, respectively.

Table 1. The main results of this paper.

Flocking type	Acceleration ($a_0(t)$)	Desired velocity ($v^0(t)$)	Desired position ($x^0(t)$)	Neighbouring graph ($\mathcal{G}_\sigma$)	Theorem	Simulation result	Example
	Conditions				Conclusions		
Strong flocking (SF) (§ 3.1)	$g_i = 1$, $\forall i \in \mathcal{I}$	$\exists k \in \mathcal{I}$ s.t. $h_s^k = h_k$ in Equation (4)	—	Connected	1–2	Figure 4	—
		$h_s^i = h_i$, $\forall i \in \mathcal{I}$ in Equation (4)	—	Connected	1–2	Figure 5	—
				Disconnected	No proof	Figure 6	
Weak flocking (WF) (§ 3.2)	$g_i = 0$, $\forall i \in \mathcal{I}$	$h_s^i = h_0$, $\forall i \in \mathcal{I}$ in Equation (12)/(15)	—	Connected	4–5	Figure 7	1(SF); 2(no SF)
			$r_0 > 0$ in Equation (15)	Connected	Brief proof	Figure 8	3(no SF)
No flocking (§ 4)	$g_i = 1$ for some $i \in \mathcal{I}$; $g_i = 0$ for others	$h_s^i = h_i$, $\forall i \in \mathcal{I}$ in Equation (4)	—	Connected	No proof	Figure 10	—
				Disconnected		Figure 11	

Note: — represents the content which is obvious and not given in detail in this paper.

These results explicitly demonstrate that the desired strong flocking motion can be obtained though the neighbouring graph varies with time. Moreover, when the coefficient h_s^i equals h_i for all $i \in \mathcal{I}_0$ in the control law (4), all agents can still move at the desired velocity eventually though the neighbouring graph is not always connected in the course of motion, as shown in Figure 6. Figures 7 and 8 show the simulation results in the case where all agents know the desired velocity but none of them know the acceleration. It is easy to see that the strong flocking motion cannot be achieved by using the control laws in (12) and (15). Hence, it is difficult for all agents to track a variable velocity $v^0(t)$ in the case where they do not know its acceleration $a_0(t)$. Figure 9 depicts the curves of the velocity errors between the agents and the CoM along x-axis and y-axis in the simulations shown in Figures 7 and 8, respectively, and from this, it is easy to see that the weak flocking motion can be obtained and the velocities of all agents converge to the velocity of the CoM. For the case where not all agents know the acceleration, we also perform some numerical simulations. Figures 10–12 consider the case where all agents know the desired velocity $v^0(t)$ but only some of them know the acceleration $a_0(t)$. The simulations show that the flocking motion cannot be achieved. This agrees with the condition on the availability of the acceleration of the virtual leader to each agent in the group as required in Theorems 1 and 2.

5. Conclusions

This paper studied the flocking problem of a group of agents moving in an n-dimensional Euclidean space with a dynamic virtual leader. To solve the problem, we proposed a set of switching control laws, and the control law acting on each agent relies on the state information of its neighbours and the external signal. We proved that, in the case where the acceleration of the virtual leader is known, all agents can follow the virtual leader, freedom from collisions between the agents is ensured, the final tight formation minimises all agent potentials, and moreover, the velocity of the CoM equals the desired velocity for all time or it will exponentially converge to the desired velocity. In the case where the acceleration is unknown, the velocities of all agents asymptotically approach the velocity of the CoM; however, in this case, the final velocity of the group may not be equal to the desired value. Numerical simulation agrees very well with the theoretical analysis.

In order to make the results more clear, we present Table 1 which includes all the cases considered in the paper.

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