*Mathematical and Computer Modelling of Dynamical Systems*
Vol. 13, No. 4, August 2007, 331–347

Taylor & Francis
Taylor & Francis Group

# The computational beauty of flocking: boids revisited

I. LEBAR BAJEC*, N. ZIMIC and M. MRAZ

University of Ljubljana, Faculty of Computer and Information Science,
Trzaska c. 25, 1000 Ljubljana, Slovenia

Artificial-life research was founded in the mid-1980s. It promotes the idea of the bottom-up research approach, where only the basic units of a situation and their local interaction are modelled, and then the system is left to evolve. However, the notable progress of the processing power of personal computers, evident in the last two decades, has had little influence on the ways the basic units (artificial animals or animats) are constructed. This impacts largely on the applicability of the methods in other research fields. Our field of choice is the modelling of bird flocks. This area was at its peak in the late 1980s when Craig W. Reynolds presented the first and most influential model – the boids. In spite of his many following works no formal definition has ever been presented. This might be the reason why a second generation of flocking models is still awaited. In this article we make a step forward, all in view of allowing for the development of the second-generation models. We present an artificial animal construction framework that has been obtained as a generalization of the existing bird flocking models, but is not limited to them. The article thus presents a formal definition of the framework and gives an example of its use. In the latter the framework is employed to present a formalization of Reynolds's boids.

*Keywords*: Animat; Artificial life; Bird; Boid; Flock; Moore automaton

*AMS Subject Classification*: 92B20; 82C22; 68T40

## 1. Introduction

Imagine a crowded street at rush hour. What makes us choose the next step when we are moving through it? Why do we decide to change sides instead of continuing in the same direction? Why do we increase our pace or slow down at the sight of a steaming croissant displayed in the window of a corner coffee shop? Our everyday life is full of such seemingly simple and yet complex decisions. We can admit that the reasons which drive our decisions are intriguing but also very difficult to simulate. It is no wonder that a substantial number of researchers have tackled this mystery. In most cases the

*Corresponding author. Email: iztok.bajec@fri.uni-lj.si

*Mathematical and Computer Modelling of Dynamical Systems*
ISSN 1387-3954 print/ISSN 1744-5051 online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/13873950600883485

authors are dealing with the simulation of animal behaviour. Such artificial animals are usually called animats – a term first introduced by Stewart W. Wilson [1].

One of the researchers who has worked on the construction of artificial animals is Craig W. Reynolds [2 – 5]. His primary interest was birds, more specifically a believable simulation of bird flocking. In [2] he defined a set of three steering behaviours, presented in table 1, that allow for a credible simulation of the flocking behaviour of birds. On the basis of these three forces every artificial bird chooses its new flight velocity (i.e. heading and speed) that allows it to be in the flock. Reynolds called his artificial bird a *boid*. Observing the behaviour of a group of boids moving through an environment, a strong resemblance to the characteristic behaviour of a flock of birds can be sensed. In addition, Reynolds states that his algorithm applies equally well to the simulation of herds and schools [2].

Reynolds translated the three drives to a set of geometrical equations, where he interpreted the expression 'nearby flockmates' as the boid's immediate surroundings. Actually, he found that a boid does not require full knowledge about the position and velocity of every boid in the flock, but only a small subset. The expression 'nearby flockmates', used in the descriptions of the steering behaviours, thus addresses the boid's awareness of another boid and is based on the distance and direction of the offset vector between them. In other words: the boid has a localized perception of the world with a certain perception distance and field of view and can be visualized as a perception volume shaped like a sphere with a cone removed from the back. It is important to note that, when the boids are in a flock, the individual perception volumes overlap and each individual boid will probably end up in a number of perception volumes.

The collision avoidance and velocity matching steering behaviours are complementary and together represent one motivation – the desire to avoid collisions within the flock. Reynolds addresses them as *static* (collision avoidance) and *dynamic* (velocity matching) [2]. The primary reason for doing so is his implementation of the two steering behaviours. He has based collision avoidance purely on the relative positions of the observed boid's flockmates and ignores their velocity. Conversely he has based velocity matching purely on velocity but ignores the relative position. If the boids are doing a good job at velocity matching, it is unlikely that a collision will occur in the near future, since all of them are flying with the same heading and speed. Therefore velocity matching is in fact a predictive version of collision avoidance. Reynolds explains that collision avoidance serves to establish the minimum required separation distance; velocity matching tends to maintain it [2]. On the other hand, the flock centring steering behaviour is the boid's urge to be a member of the flock. It is manifested by the boid's tendency to fly into the centre of the flock. Since the boid's perception is localized, this means the centre of its nearby flockmates. This tendency is the result of the boid's motivation to be evenly surrounded by its flockmates. Reynolds also states that flock centring allows for the bifurcation of simulated flocks [2]. When moving through an environment with obstacles, a real flock bifurcates and later rejoins in order to avoid possible collisions with an obstacle. The latter is one of the most admired aspects of natural flocks.

Table 1. Reynolds's steering behaviours.

| Steering behaviour | Description |
|---|---|
| Collision avoidance | Avoid collision with nearby flockmates |
| Velocity matching | Attempt to match velocity with nearby flockmates |
| Flock centring | Attempt to stay close to nearby flockmates |

Usually the knowledge about the behaviour of birds or animals in general is available only in the form of a linguistic description, which was the case even for Reynolds's boids. A transition from a linguistic description to a mathematical model is thus required and in most cases it is far from being straightforward. There are research fields, however, where exact knowledge about the behaviour of the studied phenomena is a non-affordable luxury. This is why our future research focuses on the simplification of this transition. We find the bottom-up approach as the right way to go, the one that could be used from ethologists studying animal behaviour to researchers working in the field of nanotechnology. Our leading hypothesis is that by simplifying the transition from a linguistic description to a mathematical model, a larger number of researchers would find out the remarkable possibilities of the bottom-up approach.

In the following article we present a reformulation of Wilson's animat [1] that allows the construction of artificial animals. To show its effectiveness, we use Reynolds's boid [2] as an example where we limit ourselves to dull environments, leaving the exciting ones for future research. In section 2 we start by presenting the formal definition of the artificial animal construction framework. We continue by presenting our case study, where in section 3 we use the framework to present a formalizations of Reynolds's [2] model. In section 4, using Reynolds's boids as an example, we show how the behaviour of a group of animats could be analysed.

## 2. Modelling an artificial animal

Let us consider that we are modelling the behaviour of a living being – be it a plant, an animal or a human. In the rest of this article we shall, for reasons of brevity, refer to all simply by the name *animal*. According to Wilson [1], if we want to model the behaviour of an animal with a satisfactory accuracy we need to abstract its basic characteristics:

(i) an animal exists in a sea of sensory signals, where at any moment only some signals are significant, the rest are irrelevant;
(ii) the animal is capable of actions (e.g. movement) which tend to change these signals;
(iii) certain signals (e.g. those attendant upon consumption of food), or the absence of certain signals (e.g. absence of pain) have special status for it; and
(iv) the animal acts, both externally and through internal operations, so as to approximately optimize the rate of occurrence of the special signals.

The animal's sensory-motor situation is described in very general terms by characteristics (i) and (ii). Characteristics (iii) and (iv) are assumptions which provide a way of making the notion of 'needs' (or drives) and their satisfaction. These characteristics suggest that an animal acts as an automaton and together they form the basis of Wilson's artificial animal named simply an *animat* [1].

The four characteristics represent one of the more widely adopted theories about the behaviour of animals, where every action is the result of the perception of certain signals existing in the environment and satisfaction of personal drives or goals (figure 1). We shall therefore adopt Wilson's idea and model our animat construction framework with a finite state automaton. This section presents its formal definition.

For a better understanding of the underlying theory we first present the formal definition of the Moore automaton [6] and then progress to the formal definition of the animat.

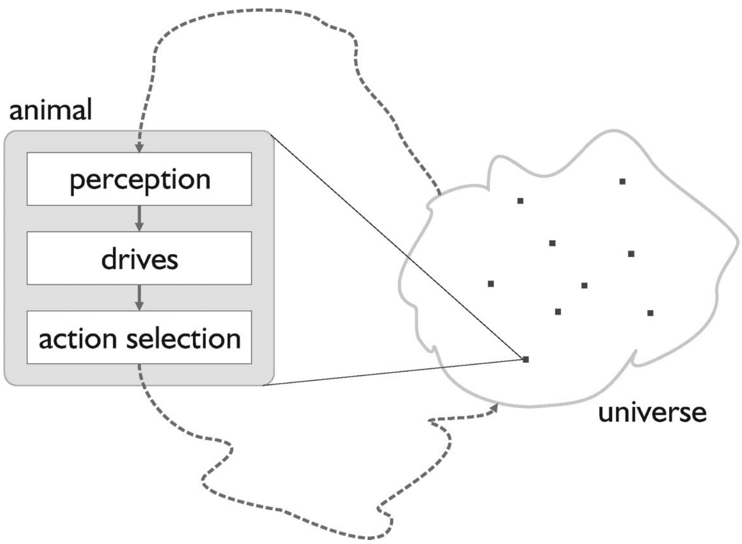


Figure 1. The three stages of the adopted theory about the behaviour of animals.

**Definition 2.1.** The Moore automaton is defined as a five-tuple $\langle \mathbb{X}, \mathbb{Q}, \mathbb{Y}, \delta, \lambda \rangle$, where $\mathbb{X}$, $\mathbb{Q}$ and $\mathbb{Y}$ are non-empty sets representing the input alphabet, the internal states and the output alphabet respectively; $\delta$ is a mapping called the transition function and $\lambda$ is a mapping called the output function:

$$\delta : \mathbb{X} \times \mathbb{Q} \rightarrow \mathbb{Q}, \tag{1}$$

$$\lambda : \mathbb{Q} \rightarrow \mathbb{Y}. \tag{2}$$

If $\mathbb{T}$ is a finite non-empty set of discrete time steps, then at a discrete time step $t \in \mathbb{T}$ the automaton is in a state $q(t) \in \mathbb{Q}$ emitting the output $\lambda(q(t)) \in \mathbb{Y}$. If an input $x(t) \in \mathbb{X}$ is applied at this time step, then, in the next discrete time step $t+1$, the automaton assumes a new state $q(t+1) = \delta(x(t), q(t))$ and starts emitting a new output $\lambda(q(t+1))$.

It soon becomes evident that modelling perception, drives and action selection with a Moore automaton is far from being straightforward. We shall thus reformulate the transition function $\delta$ from Definition 2.1 into a three-stage scheme that takes into account the adopted theory about animal behaviour (figure 1).

Let us assume that Moore automata are used to represent inanimate and/or animate objects existing in the universe. In other words, we are assuming that the output of a participating automaton represents data about an animal (e.g. position, sex, emotion, etc.) that can be perceived (e.g. in real life through physical appearance, smell, pose, etc.) by an outside observer. Any participating automaton thus, if it is to represent an animal, needs to be able to perceive this data. Therefore all of them use the same input alphabet; the Cartesian product of the output alphabets of the participating automata.

If $A_1, \ldots, A_n$ are used to denote the participating automata, then, at a discrete time step $t \in \mathbb{T}$, the state of the universe is given by the $n$-tuple $u(t) = \langle \lambda_1(t), \ldots, \lambda_n(t) \rangle$, where for all $i = 1, \ldots, n$ $\lambda_i(t)$ denotes the output of automaton $A_i$ at time step $t$. In addition, the input that is at time step $t \in \mathbb{T}$ applied to all participating automata is $x(t) = u(t)$, and thus the input alphabet employed by all participating automata is $\mathbb{X} = \mathbb{Y}_1 \times \cdots \times \mathbb{Y}_n$, where $\mathbb{Y}_i$ is the output alphabet of automaton $A_i$.

From the above discussion it can be concluded that the state of the universe is actually defined by the perceivable data about the animate and/or inanimate objects that exist in it. Perception can afterwords be thought of as a process of interpreting this data and selecting just the relevant information from all of the sensory signals (e.g. detect the locations of food sources in the vicinity). The latter can be treated as characteristic (i) of Wilson's animat [1]. In real life there exist multiple perception types (e.g. sight, smell, etc.), where each of them selects only the information that is relevant according to the specific characteristics of its type. Interpretation can be modelled as a mapping from perceivable data to information, whereas selection as the construction of an index set containing only the indexes of relevant perceivable data.

Let $q \in \mathbb{Q}$ be the current state of the observed automaton and let its input be $x = \langle \lambda_1, \ldots, \lambda_n \rangle$, where $\lambda_i$ is the current output of automaton $A_i$. For all $i = 1, \ldots, n$ let the set $\mathbb{I}_i^c$ represent information regarding characteristic c that can be obtained about automaton $A_i$ from $\lambda_i$. Information regarding characteristic c (i.e. interpreted data), obtained from the current state of the universe, can then be represented as $o \in \mathbb{I}_1^c \times \cdots \times \mathbb{I}_n^c$. Let $\mathbb{N}_n$ denote the set of all positive natural numbers less than or equal to $n$. Then the set $\mathbb{N} \in \mathcal{P}(\mathbb{N}_n)$ represents the set of indexes of members of $o$ that are, according to the specific characteristic, relevant to the observed automaton. The ordered pair $\langle \mathbb{N}, o \rangle$, where $\mathbb{N} \in \mathcal{P}(\mathbb{N}_n)$ and $o \in \mathbb{I}_1^c \times \cdots \times \mathbb{I}_n^c$, thus represents information regarding the characteristic c that was obtained from the current state of the universe and is, according to this specific characteristic, relevant to the observed automaton. For reasons of notational simplicity we shall denote the set $\mathcal{P}(\mathbb{N}_n) \times (\mathbb{I}_1^c \times \cdots \times \mathbb{I}_n^c)$ simply as $\mathbb{P}^c$.

**Definition 2.2.** Let $x \in \mathbb{X}$ be the current state of the universe and $p \in \mathbb{P}^c$ be the information regarding characteristic c that was obtained from $x$ and is, according to this characteristic, relevant to the observed automaton. Then a *perception function* for characteristic c is a mapping $P : \mathbb{X} \times \mathbb{Q} \mapsto \mathbb{P}^c$.

For simplicity, as well as to indicate the relation to cellular automata [7], we shall address the image of a perception function with the name *neighbourhood*. A neighbourhood represents only the relevant and characterized sensory signals. This means that the perception function allows us to take into account that different animals employ different strategies for sampling sensory data [8].

By following Wilson's [1] interpretation of the personal goals of an animal (animat characteristics (iii) and (iv)), it can be concluded that the animal's drives are in strong correlation of its internal state and the state of the universe. The animal's actions thus depend on the information obtained from the universe and its current internal state.

**Definition 2.3.** Let there be $k$ neighbourhoods that were obtained by means of $k$ perception functions and let $\mathbb{P}$ denote the set $\mathbb{P}^{c_1} \times \cdots \times \mathbb{P}^{c_k}$. Let $a \in \mathbb{A}$ be an action that, with respect to the neighbourhoods $\langle p_1, \ldots, p_k \rangle \in \mathbb{P}$ and $q \in \mathbb{Q}$, satisfies a specific drive. Then a *drive function* is a mapping $D : \mathbb{P} \times \mathbb{Q} \mapsto \mathbb{A}$.

Since an animal usually has more than one drive, the resulting actions must be combined so as to approximately optimize the satisfaction of all drives (animat characteristic (iv)). Not simply combined, the actions must be prioritized and, in case of conflicts, also arbitrated. Moreover, their urgency must also be taken into account.

**Definition 2.4.** Let there be $l$ actions that were obtained by means of $l$ drive functions, which satisfy $l$ drives. Then an *action selection function* is a mapping $S : \mathbb{A}^l \times \mathbb{Q} \rightarrow \mathbb{Q}$.

The animat can therefore be defined as an extended Moore automaton.

**Definition 2.5.** An animat $A = \langle \mathbb{X}, \mathbb{Q}, \mathbb{Y}, \delta, \lambda, P, D, S \rangle$ is a special Moore automaton, where $P = \langle P_1, \ldots, P_k \rangle$ is a $k$-tuple of perception functions, $D = \langle D_1, \ldots, D_l \rangle$ is an $l$-tuple of drive functions, $S$ is an action selection function and the transition function $\delta$ is defined as:

$$\delta(x, q) = S(\langle a_1, \ldots, a_l \rangle, q) \tag{3}$$

$$a_j = D_j(\langle p_1, \ldots, p_k \rangle, q), \; j = 1, \ldots, l \tag{4}$$

$$p_i = P_i(x, q), \; i = 1, \ldots, k. \tag{5}$$

Let us sum up: at a discrete time step $t \in \mathbb{T}$, an animat is input the current perceivable state of the universe (i.e. an $n$-tuple containing the current outputs of the animats that constitute the universe). The three stage scheme of the transition function $\delta$, given by equations (3)–(5), imitates the adopted theory about the behaviour of animals (figure 1). The first stage employs the perception functions $P_i$ ($i = 1, \ldots, k$) to retrieve from the current perceivable state of the universe only the information that is relevant to the observed animat. The second stage employs the drive functions $D_j$ ($j = 1, \ldots, l$) to decide, based on the retrieved information, on the actions that will satisfy the observed animat's needs. The third stage employs the action selection function $S$ to combine, prioritize and arbitrate between the potentially conflicting actions and generate the observed animat's next discrete time step state $q(t+1)$.

## 3. Formalization of Reynolds's boids

In the previous section we presented a formal definition of an artificial animal construction framework. In this section we are going to show that a boid [2] is an animat, and present its formal definition. Reynolds has based all of the boid's processing on geometrical expressions, but even though he has published numerous works concerning the matter [2–5], no formal definitions have ever been given. Recently, however, he made available the OpenSteer library,* which includes an implementation of the boids. The assumptions employed in the formalization of the model are based on this implementation.

Reynolds [2] models the boid as an object existing in a universe that is inhabited by identical boids. In some of his studies [2–4] he did introduce environmental obstacles, but for reasons of simplicity we concentrate on dull universes (i.e. no environmental obstacles).

*OpenSteer Library. Available online at: opensteer.sourceforge.net (accessed 25 May 2006).

Let us assume that a boid is an animat. We thus need to define the set of internal states, the output alphabet, the perception functions, drive functions and the corresponding action selection function. Let us begin with the set of internal states. A boid moves through the universe with a certain velocity (i.e. heading and speed). This means that the boid's internal state consists of at least its position $\boldsymbol{p} \in \mathbb{V}$, where $\mathbb{V} = \mathbb{R}^d$, $(d = 2, 3)$ is a Euclidean vector space, and velocity $\boldsymbol{v} \in \mathbb{V}$. The velocity vector $\boldsymbol{v}$ gives the boid's relative position changes per coordinate axis in the Cartesian coordinate system and therefore codes the boid's heading and speed. According to table 1, the boid has three drives and each drive is based on the expression 'nearby flockmates'. However, as discussed in the introduction, this expression has different meanings, depending on the drive. Based on Reynolds's latest implementation of the model, the latter can be interpreted as different perception volumes, defined respectively by a perception distance and a field of view. Therefore we define the separation $r_s$, alignment $r_a$ and cohesion $r_c$ perception distance, as well as the separation $fov_s$, alignment $fov_a$ and cohesion $fov_c$ field of view, which together define the corresponding perception volumes. Additionally, the boid's physics is governed by the so-called *geometric flight* [2], which imposes the following constraints: a maximal achievable speed $v_M$, which represents a simple model of viscous speed damping (i.e. the inability to exceed a certain speed even if constantly accelerating), and a maximal available force $f_M$, which takes into account the fact of modelling an animal with a finite amount of energy. Furthermore, as Reynolds's model is based on Newtonian forces, the boid also has a mass $m$. For reasons of simplicity we shall assume that only the boid's position $\boldsymbol{p}$ and velocity $\boldsymbol{v}$ can change through time, while all other internal state parameters stay constant. The latter is consistent with the original boid definition [2].

**Definition 3.1.** Let $\mathbb{V} = \mathbb{R}^d$, $(d = 2, 3)$ be a Euclidean vector space, then the boid's internal state $q \in \mathbb{Q}$ is defined by (6), where $\boldsymbol{p} \in \mathbb{V}$ is its position, $\boldsymbol{v} \in \mathbb{V}$ is its velocity, $r = \langle r_s, r_a, r_c \rangle$ are the separation, alignment and cohesion perception distance, $fov = \langle fov_s, fov_a, fov_c \rangle$ are the separation, alignment and cohesion field of view, $m$ is its mass, $v_M$ is its maximal achievable speed and $f_M$ is its maximal available force.

$$\mathbb{Q} = \{q \mid q = \langle \boldsymbol{p}, \boldsymbol{v}, r, fov, m, v_M, f_M \rangle\} \tag{6}$$

In every discrete time step the boid changes its velocity in order to approximately optimize a certain set of drives (i.e. the three steering behaviours presented in table 1). The decision is based purely on its current state and the current state of the universe. More precisely, the decision is based on the perceived locations and velocities of its nearby flockmates. Therefore at any given time step a boid makes available only the data about its position $\boldsymbol{p}$ and velocity $\boldsymbol{v}$. In other words the boid's output alphabet is $\mathbb{Y} = \mathbb{V} \times \mathbb{V}$ and its output is defined as $\lambda(q) = \langle \boldsymbol{p}, \boldsymbol{v} \rangle$.

As already said, a perception function interprets the state of the universe and selects only relevant perceivable data. The perception model employed by Reynolds can be visualized as a perception volume shaped like a sphere with a cone removed from the back. The latter can be defined by means of a perception distance and field of view. Since the state of the universe is represented by a collection of pairs $\langle \boldsymbol{p}, \boldsymbol{v} \rangle$, locations and velocities of the participating boids, the interpretation is simple and the set representing information regarding the location and velocity of a boid is defined as $\mathbb{I} = \mathbb{Y}$. The three perception functions thus differ only in the sets that represent the indexes of relevant information. In the case of the separation perception this set represents the indexes of the flockmates that must be avoided, in the case of the alignment perception indexes of the flockmates that must be followed, and in the case of the cohesion perception the flockmates that must be kept close to.

Let $B_j$ and $B_i$ be two boids and let $q_j$ and $q_i$ be their corresponding internal states. Let $\lambda_i = \lambda(q_i) = \langle \boldsymbol{p}_i, \boldsymbol{v}_i \rangle$ be the output of boid $B_i$ and let $B_j$ denote the observed boid. The distance of boid $B_i$ from the observed boid $B_j$ is then defined as

$$\Delta(\lambda_i, q_j) = \left\| \boldsymbol{p}_i - \boldsymbol{p}_j \right\|, \tag{7}$$

and its angular offset is then defined as

$$\varphi(\lambda_i, q_j) = \arccos\left( \frac{\boldsymbol{v}_j \cdot (\boldsymbol{p}_i - \boldsymbol{p}_j)}{\|\boldsymbol{v}_j\| \|\boldsymbol{p}_i - \boldsymbol{p}_j\|} \right). \tag{8}$$

**Definition 3.2.** Let $x = \langle \lambda_1, \ldots, \lambda_n \rangle$ be the state of the universe and $\mathbb{I} = \mathbb{Y}$ the set representing information about a flockmate's location and velocity. Let $j$ be the index of the observed boid. Then $\mathbb{P}^{\mathrm{v}} = \mathcal{P}(\mathbb{N}_n) \times \mathbb{I}^n$ and the perception function $P_{\mathrm{s}} : \mathbb{X} \times \mathbb{Q} \mapsto \mathbb{P}^{\mathrm{v}}$ defined by equation (9) is the *separation perception* function (i.e. perception of nearby flockmates that must be avoided), the perception function $P_{\mathrm{a}} : \mathbb{X} \times \mathbb{Q} \mapsto \mathbb{P}^{\mathrm{v}}$ defined by equation (10) is the *alignment perception* function (i.e. perception of nearby flockmates with which to match velocity), and the perception function $P_{\mathrm{c}} : \mathbb{X} \times \mathbb{Q} \mapsto \mathbb{P}^{\mathrm{v}}$ defined by equation (11) is the *cohesion perception* function (i.e. perception of nearby flockmates that must be kept close to).

$$\begin{gathered} P_{\mathrm{s}}(x, q) = \langle \mathbb{N}_{\mathrm{s}}, x \rangle, \\ \mathbb{N}_{\mathrm{s}} = \{ i | i \in \mathbb{N}_n, i \neq j, \Delta(\lambda_i, q_j) \leq r_{\mathrm{s}}, \varphi(\lambda_i, q_j) < fov_{\mathrm{s}} \} \end{gathered} \tag{9}$$

$$\begin{gathered} P_{\mathrm{a}}(x, q) = \langle \mathbb{N}_{\mathrm{a}}, x \rangle, \\ \mathbb{N}_{\mathrm{a}} = \{ i | i \in \mathbb{N}_n, i \neq j, \Delta(\lambda_i, q_j) \leq r_{\mathrm{a}}, \varphi(\lambda_i, q_j) < fov_{\mathrm{a}} \} \end{gathered} \tag{10}$$

$$\begin{gathered} P_{\mathrm{c}}(x, q) = \langle \mathbb{N}_{\mathrm{s}}, x \rangle, \\ \mathbb{N}_{\mathrm{c}} = \{ i | i \in \mathbb{N}_n, i \neq j, \Delta(\lambda_i, q_j) \leq r_{\mathrm{s}}, \varphi(\lambda_i, q_j) < fov_{\mathrm{c}} \}. \end{gathered} \tag{11}$$

The boid's behaviour is based on the three steering behaviours presented in table 1 (see [2,4] for detailed descriptions). Reynolds [2,4] modelled them using geometrical calculations. Each steering behaviour thus produces a force that would induce a change in velocity that satisfies the corresponding need (i.e. of avoiding collisions, of matching velocity, and of being part of a flock). A drive function (Definition 2.3) returns an action that will satisfy a specific need. In other words: if we define the set of actions to be a Euclidean vector space and an action to be the required force, then we can use Reynolds's geometrical calculations to define the *separation*, *alignment* and *cohesion* drive.

Separation represents the boid's need of a personal space and is a mathematical approximation of the collision avoidance steering behaviour (table 1). The main idea behind it is that it keeps the boids free from collisions. Alignment represents the boid's wish to move with the same velocity as its flockmates. It is a mathematical approximation of the velocity matching steering behaviour (table 1). The main idea is that when moving with the same velocity (i.e. with the same heading and with the same speed), the boids will not collide with each other in the near future. Finally, cohesion represents the boid's wish to be a member of the flock. It is a mathematical approximation of the flock centring steering behaviour (table 1). The main idea is that every boid wants to be evenly surrounded by its flockmates. Thus if the boid senses the presence of flockmates on just one of its sides it moves to that side. Indeed, Pliny [9] was one of the first who noted that 'it is a peculiarity of the starling kind that they fly in flocks and wheel round in a sort of circular ball, all making towards the centre of the flock'.

Let $\boldsymbol{v} \in \mathbb{V}$ be a vector. Then the normalized vector $\boldsymbol{v}$ (i.e. a vector in the same direction as $\boldsymbol{v}$, but with size 1) is defined as

$$\boldsymbol{v}^0 = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}. \tag{12}$$

**Definition 3.3.** Let there be three perception functions $P_{\mathrm{s}}$, $P_{\mathrm{a}}$ and $P_{\mathrm{c}}$ (Definition 3.2) and let $p_{\mathrm{s}}$, $p_{\mathrm{a}}$ and $p_{\mathrm{c}}$ be the corresponding neighbourhoods. Then $p_{\mathrm{s}} = \langle \mathbb{N}_{\mathrm{s}}, o \rangle$, $p_{\mathrm{a}} = \langle \mathbb{N}_{\mathrm{a}}, o \rangle$, and $p_{\mathrm{c}} = \langle \mathbb{N}_{\mathrm{c}}, o \rangle$, where $o = \langle \lambda_1, \ldots, \lambda_n \rangle$ and $\forall i = 1, \ldots, n \; \lambda_i = \langle \boldsymbol{p}_i, \boldsymbol{v}_i \rangle$. Let $\mathbb{P} = (\mathbb{P}^{\mathrm{v}})^3$ and let $p \in \mathbb{P}$ be $p = \langle p_{\mathrm{s}}, p_{\mathrm{a}}, p_{\mathrm{c}} \rangle$. Let the set of available actions be $\mathbb{A} = \mathbb{V}$. Then the drive function $D_{\mathrm{s}} : \mathbb{P} \times \mathbb{Q} \mapsto \mathbb{A}$ defined by equation (13) is the *separation drive* function, the drive function $D_{\mathrm{a}} : \mathbb{P} \times \mathbb{Q} \mapsto \mathbb{A}$ defined by equation (14) is the *alignment drive* function, and the drive function $D_{\mathrm{c}} : \mathbb{P} \times \mathbb{Q} \mapsto \mathbb{A}$ defined by equation (15) is the *cohesion drive* function.

$$D_{\mathrm{s}}(p, q) = \left[ \sum_{i \in \mathbb{N}_{\mathrm{s}}} \frac{\boldsymbol{p} - \boldsymbol{p}_i}{\|\boldsymbol{p}_i - \boldsymbol{p}\|^2} \right]^0 \tag{13}$$

$$D_{\mathrm{a}}(p, q) = \left[ \left( \frac{1}{|\mathbb{N}_{\mathrm{a}}|} \sum_{i \in \mathbb{N}_{\mathrm{a}}} \boldsymbol{v}_i \right) - \boldsymbol{v} \right]^0 \tag{14}$$

$$D_{\mathrm{c}}(p, q) = \left[ \left( \frac{1}{|\mathbb{N}_{\mathrm{c}}|} \sum_{i \in \mathbb{N}_{\mathrm{c}}} \boldsymbol{p}_i \right) - \boldsymbol{p} \right]^0 \tag{15}$$

The boid's perception of the universe was defined by using perception functions whereas its steering behaviours were defined by using drive functions. Each drive function returns an action (i.e. a force) that would induce a velocity change that satisfies a specific need (see table 1). To compute the action that would satisfy all of the needs the resulting actions (forces) have to be combined. But, as in some cases (e.g. two flocks joining) the latter can be contradictory; they must be prioritized and also arbitrated. In his original study Reynolds [2] proposes combining them using a special algorithm named 'prioritized acceleration allocation'. In a later study, however, he admits that in the course of several re-implementations of the model over the years, a simpler linear combination has proved sufficient [4]. This approach is used even in his latest implementation of the OpenSteer Library and thus also the approach employed by us.

Let $\boldsymbol{v} \in \mathbb{V}$ be a vector and let $v \in \mathbb{R}^+$ be its maximal size. Then the truncation of vector $\boldsymbol{v}$ is defined as

$$\lfloor \boldsymbol{v} \rceil^{v} = \begin{cases} \boldsymbol{v} & \text{iff } \|\boldsymbol{v}\| \leq v \\ v\boldsymbol{v}^0 & \text{iff } \|\boldsymbol{v}\| > v. \end{cases} \tag{16}$$

**Definition 3.4.** Let there be three drive functions $D_\mathrm{s}$, $D_\mathrm{a}$ and $D_\mathrm{c}$ (Definition 3.3) and let $\boldsymbol{a}_\mathrm{s}$, $\boldsymbol{a}_\mathrm{a}$ and $\boldsymbol{a}_\mathrm{c}$ be the corresponding actions. Then $a \in \mathbb{A}^3$ is $a = \langle \boldsymbol{a}_\mathrm{s}, \boldsymbol{a}_\mathrm{a}, \boldsymbol{a}_\mathrm{c} \rangle$. Let $w_\mathrm{s}$, $w_\mathrm{a}$ and $w_\mathrm{c}$ represent the significance of action $\boldsymbol{a}_\mathrm{s}$, $\boldsymbol{a}_\mathrm{a}$ and $\boldsymbol{a}_\mathrm{c}$ respectively. Then the action selection function $S_\mathrm{ws} : \mathbb{A} \times \mathbb{Q} \mapsto \mathbb{Q}$, defined by equation (17), is the *weighted sum action selection* function.

$$\begin{aligned} S_\mathrm{ws}(a, q) &= \langle \boldsymbol{p}', \boldsymbol{v}', r, fov, m, v_\mathrm{M}, f_\mathrm{M} \rangle \\ \boldsymbol{v}' &= \left\lfloor \boldsymbol{v} + \frac{\lfloor w_\mathrm{s}\boldsymbol{a}_\mathrm{s} + w_a\boldsymbol{a}_\mathrm{a} + w_\mathrm{c}\boldsymbol{a}_\mathrm{c} \rceil^{f_\mathrm{M}}}{m} \right\rceil^{v_\mathrm{M}} \\ \boldsymbol{p}' &= \boldsymbol{p} + \boldsymbol{v}'. \end{aligned} \tag{17}$$

By employing Definitions 3.1 – 3.4 Reynolds's boid can be defined as a special animat.

**Definition 3.5.** A boid $\mathrm{B} = \langle \mathbb{X}, \mathbb{Q}, \mathbb{Y}, \delta, \lambda, P, D, S \rangle$ is a special animat. The set of internal states $\mathbb{Q}$ is defined by Definition 3.1, the output alphabet is $\mathbb{Y} = \mathbb{V} \times \mathbb{V}$ and the output function is $\lambda(q) = \langle \boldsymbol{p}, \boldsymbol{v} \rangle$. The input alphabet is $\mathbb{X} = \mathbb{Y}^n$, where $n$ is the number of boids that consist the universe. Finally the $k$-tuple of perception functions is $P = \langle P_\mathrm{s}, P_\mathrm{a}, P_\mathrm{c} \rangle$ (Definition 3.2), the $l$-tuple of drive functions is $D = \langle D_\mathrm{s}, D_\mathrm{a}, D_\mathrm{c} \rangle$ (Definition 3.3) and the action selection function is $S = S_\mathrm{ws}$ (Definition 3.4).

## 4. Behaviour analysis

In the previous two sections we presented the formal definition of an artificial animal construction framework and then employed it to present a formalization of a boid. The latter is a computer model of a bird, developed by Reynolds in the late 1980s [2]. As the model purports to simulate bird flocking behaviour, this section will focus to the quality of the flocking behaviour of the formalized model. First a set of metrics, with which one can measure and judge the flocking behaviour of a group of boids, will be presented and then the latter will be used in a series of controlled experiments to evaluate the flocking behaviour of boids. For reasons of (presentational) simplicity the experiments will be constrained to a two-dimensional universe.

As Reynolds's [2] main objective is to artificially simulate the characteristic behaviour of a flock of birds, the main question that arises is, '*What is a flock*'? As it turns out, Reynolds with this term refers to a group of objects that exhibit a general class of *polarized*, *non-colliding*, *aggregate motion*, where the term *polarization* is from zoology, meaning 'alignment' of animal groups [2]. However, from an ornithological point of view, a flock is a group of flying birds coordinated in one or more of the following parameters of flight: turning, spacing, speed and heading of individual birds, and time of takeoff and landing [10]. Neither definition, nevertheless, gives enough information to allow for an algorithmic classification. Let us thus define our interpretation of a flock (figure 2).

Let the universe consist of $n$ boids and let us denote them as $B_1, \ldots, B_n$. Let the current perceivable state of the universe be $u = \langle \lambda_1, \ldots, \lambda_n \rangle$, where $\lambda_i = \langle \boldsymbol{p}_i, \boldsymbol{v}_i \rangle$ denotes the current position and velocity of boid $B_i$, for all $i = 1, \ldots, n$. Let $r_I$ denote the maximal distance between two boids that still allows for a direct influence between them. Recalling from Definitions 3.1 and 3.3: a boid has three distinct perception volumes, each defined through a perception distance and a field of view. Considering only the perception distance, two boids can be in range to potentially end up in at least one of each other's perception volumes, however due to their orientation they are actually not in any of them (e.g. see 'leader' in figure 2). To conclude: since $r_s$, $r_a$ and $r_c$ denote the separation, alignment and cohesion perception distances and since all boids are created equal, then in the boid's case the range of potential influence is $r_I = \max(r_s, r_a, r_c)$.

Thus, based on the current perceivable state of the universe $u$ and the range of potential influence $r_I$, the set $\mathbb{M} \subseteq \mathbb{Y}^n \times \mathbb{Y}^n$, which represents the relation of potential direct influence between boids can be computed:

$$\mathbb{M} = \left\{(i, j) |\ i, j \in \mathbb{N}_n, j \neq i, \|\boldsymbol{p}_j - \boldsymbol{p}_i\| \leq r_I \right\}. \tag{18}$$

The relation of potential direct influence is in fact a set of ordered index pairs $(i, j)$, where boid $B_j$ is close enough to be treated as relevant by at least one of the perception functions of boid $B_i$. This means that by traversing this list we can find out which boids potentially directly or indirectly influence one another.

Let $b = b_0 \ldots b_m$, where $b_i \in \mathbb{N}_n$, $(i = 1, \ldots, m)$, denote a series of indexes. The set representing the relation of potential direct or indirect influence is then

$$\mathbb{M}^\star = \{(i, k) |\ i, k \in \mathbb{N}_n, \exists b (b_0 = i, b_m = k, (b_{j-1}, b_j) \in \mathbb{M}, \forall j \in \mathbb{N}_m)\}. \tag{19}$$

Let $\mathbb{G}_i$ denote the set of indexes of boids that directly or indirectly influence boid $B_i$. Then $\mathbb{G}_i$ is defined as

$$\mathbb{G}_i = \{j \,|\, j \in \mathbb{N}_n, (i, j) \in \mathbb{M}^\star\}. \tag{20}$$

Let a straggler be a boid that does not and is not influenced by any other boid and let a flock be a set of boids that potentially influence one another. Then the set of stragglers is defined as

$$\mathbb{S} = \{\mathbb{G}_i \,|\, i \in \mathbb{N}_n, |\mathbb{G}_i| = 0\}, \tag{21}$$

and the set of flocks as

$$\mathbb{F} = \{\mathbb{G}_i \,|\, i \in \mathbb{N}_n, |\mathbb{G}_i| \geq 1\}. \tag{22}$$

By definition a flock is thus a set of boids that have a potential direct or indirect influence on one another. In ornithological studies of bird flocks, the leading role was played by the search for evidence that would answer the question of existence or

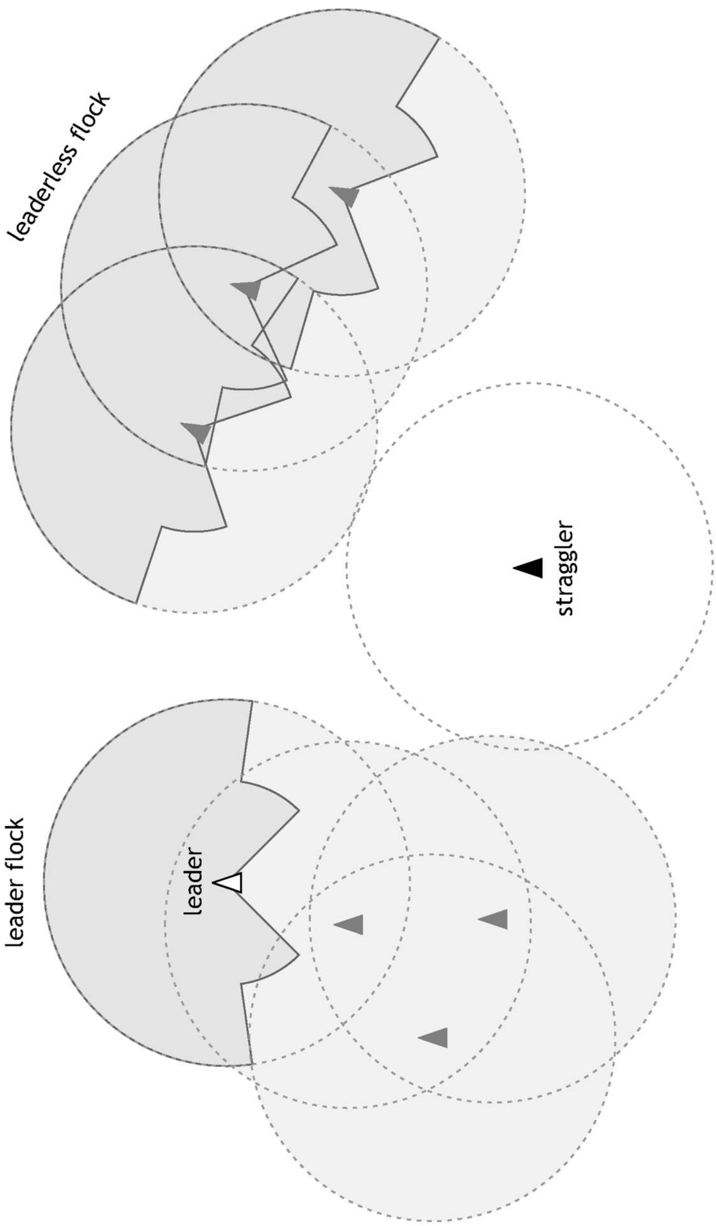


Figure 2. A leader flock, a leaderless flock and a straggler. Triangles represent boids with the apex indicating the heading. The dashed lines represent the range of potential influence. The light shaded areas thus represent the flocks' extents. The dark shaded areas are used to depict the perception volumes.

necessity of a flock leader. Indeed, many researchers assumed a leader's existence and presumed it directed the movement of the whole flock, however, efforts to identify the flock's leader have been so far unsuccessful [11–14]. In a simulation this question becomes rather simple – at any time it is possible to determine a leading boid. Let a *leader* be a boid that is a member of a flock but is not directly affected by any of its flockmates (i.e. there is at least one flockmate that is in range for potential influence, however it is not in any of its perception volumes). In other words: a leader is a boid that is a member of a flock but all of its perception functions find all members of the flock irrelevant. Nevertheless, as the leader is a member of a flock, it has a potential influence on at least one of its flockmates. Let $B_i$ be the observed boid and let the state of the universe be $u$. Then the input of boid $B_i$ is $x = u$. Let $\langle \mathbb{N}_s, o \rangle = P_s(x, q_i)$, $\langle \mathbb{N}_a, o \rangle = P_a(x, q_i)$ and $\langle \mathbb{N}_c, o \rangle = P_c(x, q_i)$ and let $\mathbb{N}_i$ denote $\mathbb{N}_s \cup \mathbb{N}_a \cup \mathbb{N}_c$. Then boid $B_i$ is a leader if and only if $|\mathbb{N}_i| = 0$. The set of leader flocks is therefore defined as

$$\mathbb{F}_L = \{\mathbb{G} | \mathbb{G} \in \mathbb{F}(\exists i \in \mathbb{G}, |\mathbb{N}_i| = 0)\}. \tag{23}$$

The behaviour of a group of boids must be estimated by comparing the resemblance of their behaviour to that seen in natural flocks. The best choice is to turn to counting the cumulative number of collisions between boids and observe the temporal dependency of the number of stragglers and the number of flocks. Since in nature collisions rarely occur, the metric of cumulative collisions is fairly important and the lower it is, the better the flocking ability. However, as our definition of a flock does not make any note on polarization and inter-individual coordination, one might end up with a swarm and not a bird flock. Indeed, in nature they both have few if any collisions, but the first is uncoordinated, while the second is highly coordinated. The best option is thus to visually inspect the flock formation (see [10] for a discussion on flock formations and flocks in general), as well as observe the temporal dependency of the average nearest neighbour distance (NND) and the heading and speed standard

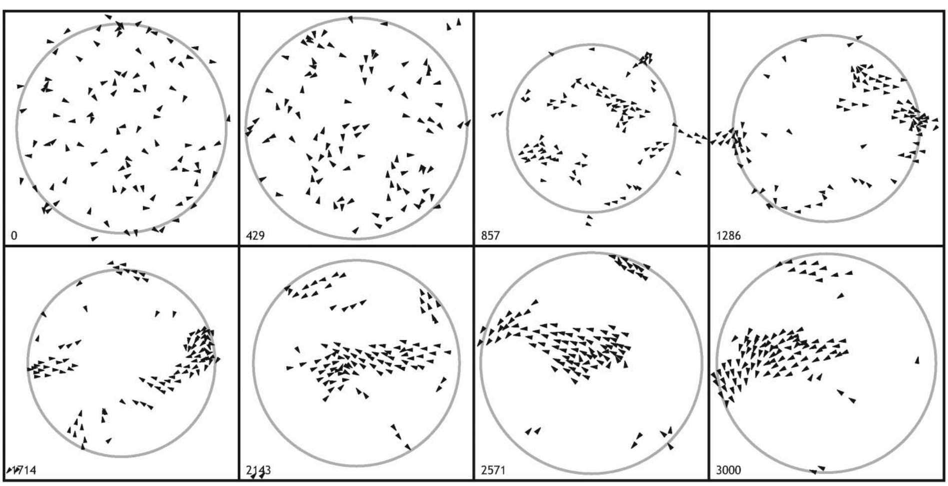


Figure 3. A series of time-equidistant frames from one of the experiments used for the estimation of flocking ability of Reynolds's boids [2]. The black triangles represent boids, with the apex indicating the heading. The grey circle surrounding the boids represents a boundary and whenever a boid crosses it, it is forced to turn and eventually return inside of it. The boids' images were scaled up to aid print clarity and thus their apparent overlapping does not necessary imply a collision.

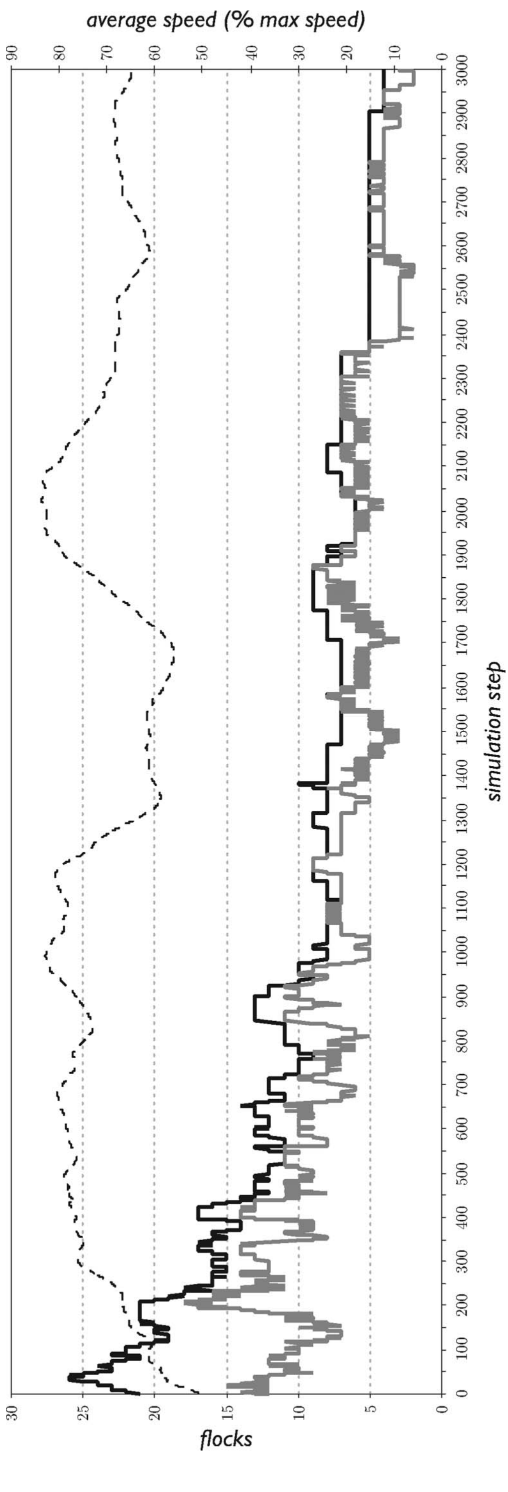


Figure 4. A chart representing the temporal dynamics of the number of flocks (black line), the contribution of leader flocks (grey line) and the temporal dynamics of the average speed (dashed line) for the same experiment as in figure 3.

deviation in every flock. As a matter of fact, the latter are the metrics that are most commonly employed by ornithologists [14–16]. Furthermore, when observing the temporal dependency of the number of flocks one should always monitor the temporal dependency of the proportion of leader flocks also.

### 4.1 *Experiments*

The flocking behaviour of the boids was evaluated by using a set of four experiments. To help boids in forming flocks they were contained within a special boundary. A boid crossing this boundary was forced to turn and eventually return inside of it, and thus the confinement actually modelled a roosting area in real flocks. The universe consisted of 100 boids that had random initial positions and velocities. The initial distribution had on average 18.5 stragglers, with a standard deviation of 4.51, and $21.5 \pm 5.26$ flocks, $61.9 \pm 5.01\%$ of which had leaders. After 3000 simulations the boids averaged $17 \pm 2.16$ collisions. Figure 3 shows a series of time-equidistant frames from one of the experiments. It was noticed that collisions were mostly head-on collision caused by the merging of flocks and occurred randomly throughout the entire simulation. In addition, in many cases, the merging caused the boids to pile-up and later to regain speed very slowly (see frames 1286–1714 and 2143–3000 in figure 3). A chart representing the temporal dynamics of the number of flocks and the temporal dynamics of the average speed is presented in Figure 4. The graphs suggest that boids do form flocks. In addition the decrease in speed caused by pile-ups is very well defined (see, for example, frames 1286–1714). After 3000 steps the average final number of stragglers was $2.75 \pm 1.5$, the average final number of flocks was $4 \pm 2.16$ and the average proportion of leader flocks observed throughout the simulations was $75.44 \pm 15.85\%$.

To estimate the influence of parameter changes on the boids' behaviour a series of eight experiments was performed. In this series, for the three weights $w_s$, $w_a$ and $w_c$ (Definition 3.4), all possible combinations of 10% and 90% of the original values employed by Reynolds have been explored. The influence was observed on the basis of one of the experiments from the first set-up. The results are summed up in table 2. It can be noticed that the separation and alignment drives have the highest impact on the number of collisions. Additionally, it can be noticed that the alignment drive has the highest influence on flock formation, however it also reduces the overall speed of

Table 2. A table summarizing the influence of parameter changes on the overall behaviour of boids. The proportion of leader flocks, average nearest neighbour distance and average speed are presented in the form of the overall average and standard deviation values computed over the whole 3000 simulation steps.

| Experiment | Collisions | Stragglers | Flocks | Proportion of leader flocks (avg) | Average NND (avg) | Average speed (avg) |
|---|---|---|---|---|---|---|
| Initial | | 21 | 21 | 38.1 | 5.42 | 51 |
| (0.1, 0.1, 0.1) | 153 | 14 | 17 | $36 \pm 14$ | $4.62 \pm 0.28$ | $49.81 \pm 1.73$ |
| (0.1, 0.1, 0.9) | 302 | 13 | 19 | $31 \pm 11$ | $4.65 \pm 0.28$ | $94.46 \pm 6.20$ |
| (0.1, 0.9, 0.1) | 59 | 6 | 4 | $49 \pm 18$ | $3.30 \pm 0.78$ | $29.24 \pm 4.76$ |
| (0.1, 0.9, 0.9) | 778 | 9 | 5 | $42 \pm 15$ | $3.05 \pm 0.56$ | $81.32 \pm 8.16$ |
| (0.9, 0.1, 0.1) | 0 | 17 | 16 | $25 \pm 11$ | $5.92 \pm 0.16$ | $39.01 \pm 4.24$ |
| (0.9, 0.1, 0.9) | 156 | 9 | 15 | $26 \pm 11$ | $5.05 \pm 0.23$ | $89.39 \pm 6.02$ |
| (0.9, 0.9, 0.1) | 0 | 8 | 7 | $24 \pm 14$ | $5.62 \pm 0.20$ | $34.48 \pm 3.78$ |
| (0.9, 0.9, 0.9) | 26 | 2 | 5 | $23 \pm 17$ | $4.36 \pm 0.37$ | $70.68 \pm 6.39$ |
| (1.0, 1.0, 1.0) | 17 | 2 | 4 | $23 \pm 15$ | $4.36 \pm 0.40$ | $70.55 \pm 7.93$ |

the flocks and nearby neighbour distance. Indeed it is surprising that, if we give the highest priority to low collision count, followed by the lowest final number of flocks and stragglers, the best behaviour is obtained when both separation and alignment drive are at 90% and only cohesion drive is at 10% of the original value.

## 5. Conclusion

We do not question the flocking behaviour of Reynolds's boids. We also understand that his primary interest was a flocking behaviour that is visually credible enough to be used in motion pictures. On the other hand, we find the artificial life approach as the right way to go when studying phenomena that are difficult to study or even cannot be studied by traditional methods. With this we address ornithologists and the numerous difficulties they face when trying to acquire and study four-dimensional data to answer the questions 'why' and 'how' birds flock [9]. As our metrics show, Reynolds's model is not so perfect from this point of view. In addition, his approach is far from being straightforward, which hinders the emergence of the second generation of bird flocking models. Indeed, he employs geometrical computation to approximate the steering behaviours. This mathematical complexity is a crucial setback for a larger number of interdisciplinary researchers to be interested in boids. With a more straightforward approach they could, for example, be used to model individual animals living in a predator–prey relation and as such represent an alternative bottom-up approach to the study of temporal development of the population densities [17]. It is likely that the employed equations *per se* are to someone unfamiliar with terms such as velocity, acceleration, etc. unusual or even discouraging. This paper therefore presents a formal definition of an artificial life construction framework, for which a transition from linguistic descriptions to mathematical approximations is currently still required, but will at a later date be omitted. Our progress on this matter can be seen in the recently published papers [18–21].

## Acknowledgements

The work presented in this paper was done at the Computer Structures and Systems Laboratory, Faculty of Computer and Information Science, University of Ljubljana, Ljubljana, Slovenia and is part of the PhD thesis being prepared by I. Lebar Bajec.

## References

[1] Wilson, S.W., 1985, Knowledge growth in an artificial animal. In: J.J. Grefenstette (Ed), *Proceedings of the First International Conference on Genetic Algorithms, Pittsburgh, PA* (Hilsdale, NJ: Lawrence Erlabum Associates), pp. 16–23.

[2] Reynolds, C.W., 1987, Flocks, herds, and schools: a distributed vehavioral model. *Computer Graphics* (SIGGRAPH 1987 Conference Proceedings), **21**, 25–34.

[3] Reynolds, C.W., 1988. Not Bumping into Things. Notes on 'obstacle avoidance' for the course on Physically Based Modeling at SIGGRAPH. Available online at: www.red3d.com/cwr/nobump/nobump.html (accessed 25 May 2006).

[4] Reynolds, C.W., 1999, Steering behaviors for autonomous characters. In: *Proceedings of the Game Developers Conference 1999, San Jose, CA* (San Francisco, CA: Miller Freeman Game Group), pp. 763–782.

[5] Reynolds, C.W., 2001, Interaction with groups of autonomous characters. In: *Proceedings of the Game Developers Conference 2001, San Jose, CA* (San Francisco, CA: CMP Game Media Group), pp. 449–460.

[6] Kohavi, Z., 1978, *Finite Automata Theory* (New York: McGraw Hill).

[7] Hayes, B., 1984, Computer recreations: the cellular automaton offers a model of the world and a world unto itself. *Scientific American*, **250**, 10–16.
[8] Cliff, D. and Bullock, S., 1993, Adding 'foveal vision' to wilson's animat. *Adaptive Behavior*, **2**, 49–72.
[9] Heppner, F.H., 1997, Three-dimensional structure and dynamics of bird flocks. In: J.K. Parrish and W.M. Hammner (Eds), *Animal Groups in Three Dimensions* (Cambridge: Cambridge University Press), pp. 68–89.
[10] Heppner, F.H., 1974, Avian Flight Formations. *Bird-Banding*, **45**, 160–169.
[11] Heppner, F.H. and Grenander, U., 1990, A stochastic nonlinear model for coordinated bird flocks. In: S. Krasner (Ed), *The Ubiquity of Chaos* (Washington, DC: AAAS), pp. 233–238.
[12] Heppner, F.H. and Hafner, J.D., 1974, Communication in bird flocks: an electro-magnetic model. In: J.G. Llaurado, A. Sances and J.H. Battocletti (Eds), *Biological and Clinical Effects of Low-Frequency Magnetic and Electric Fields* (Springfield, IL: Charles C. Thomas), pp. 147–162.
[13] Davis, J.M., 1980, The coordinated aerobatics of dunlin flocks. *Animal Behaviour*, **28**, 668–673.
[14] Pomeroy, H., and Heppner, F.H., 1992, Structure of turning in airborne rock dove (*Columba livia*) Flocks. *The Auk*, **109**, 256–267.
[15] Gould, L.L. and Heppner, F.H., 1974, The vee formation of Canada Geese. *The Auk*, **91**, 494–506.
[16] Heppner, F.H., Convissar, J.L., Moonan, D.E. and Anderson, J.G.T., 1985, Visual Angle and formation flight in Canada Geese (*Branta canadensis*). *The Auk*, **102**, 195–198.
[17] Krabs, W., 2003, A general predator–prey model. *Mathematical and Computer Modelling of Dynamical Systems*, **9**, 387–401.
[18] Lebar Bajec, I., Zimic, N. and Mraz, M., 2003, Fuzzifying the thoughts of animats. In: T. Bilgiç, B. De Baets and O. Kaynak (Eds), *Proceedings of the 10th International Fuzzy Systems Association World Congress*, Istanbul, *Turkey* (Lecture Notes in Computer Science 2715). (Heidelberg: Springer-Verlag), pp. 195–202.
[19] Lebar Bajec, I., Zimic, N. and Mraz, M., 2003, Boids with a fuzzy way of thinking. In: H. Leung (Ed), *Proceedings of the artificial intelligence and soft computing conference, Banff, Canada* (Anaheim; Calgary: ACTA Press), pp. 58–62.
[20] Mraz, M., Lebar Bajec, I., and Zimic, N., 2004, Fuzzy logic used for modelling and simulation of massive processes: a useful tool for nanotechnologies? In: M.H. Hamza (Ed), *Proceedings of Neural Networks and Computational Intelligence Conference*, *Grindelwald, Switzerland* (Zurich: ACTA Press), pp. 274–278.
[21] Lebar Bajec, I., Zimic, N. and Mraz, M., 2005, Simulating flocks on the wing: the fuzzy Approach. *Journal of Theoretical Biology*, **233**, 199–220.